Algebraic Hierarchies of cellular automata classes

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Cellular automata are a formal model of locally interacting systems. They are syntactically simple but can present extremely complex behaviors, which make them suitable to study complex systems in general. Many classifications have been proposed in literature [1], often relying on the observation of dynamics. In a first part, we present more recent approaches of algebraic nature based on notions of sub- or quotient systems. A second part is dedicated to new results concerning these algebraic tools. Actually this framework allows to set formal definitions for intuitive global notions and to prove new positive results but also, more interestingly, negative ones. More precisely, we show that modifying local rules may be more powerful in some sense than increasing the number of states; then we illustrate by the construction of an infinite lattice that dynamical universality is more powerful than usual computation universality.

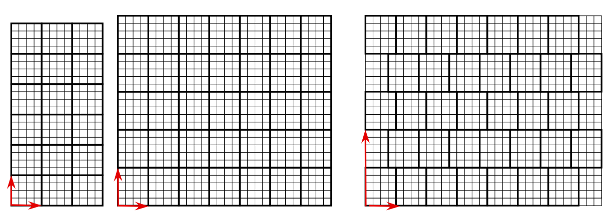

Figure 1: Squar tiling, rectangular tiling and shifted rectangular tiling.

Our approaches are to define "natural" but tractable comparison criteria of orbits (also called space-time diagrams in the case of cellular automata) and then to derive comparison criteria of sets of orbits inducing comparisons on cellular automata themselves.
Let us explain it in case of dimension 1 for sake of simplicity. The bi-dimensional underlying graphs of space-time diagrams are tiled by one (and only one) tile. Actually only bi-periodic tilings characterized by two periods - horizontal and vertical - are considered. If there is no constraint on the periods couple, one gets what will be called a "shifted rectangular tiling", if periods are equal to the sides of the tile, one gets "rectangular tilings" and if only square tiles are allowed, one gets "square tilings" (see Figure 1) [2]. Using theses tilings, the collection of space-time diagrams of a given cellular automaton can be transformed into the collection of space-time diagrams of a new cellular automaton, states of which are the obtained colored tiles. This new automaton is said to be constructed by "shifted rectangular (or rectangular, or square) grouping".

Two standard ways, according to the existence of some injection or some surjection between sets of states, allow to compare automata rules: A is said to be a sub-automaton of B [3] (resp. a quotient-automaton of B) if it is isomorphic to the restriction of B to a subset of its set of states (resp. if it is isomorphic to a cellular automaton obtained from B by identification of some states).
Then two relations can be defined over the set of cellular automata which happen to be pre-orders: $A \leq_s B$ (resp. $A \leq_q B$) if "some grouping of A is a sub-automaton (resp. a quotient-automaton) of some grouping of B. These pre-orders (actually six) induce corresponding equivalence relations and orders on the classes.

Intuitively $A \leq_s B$ means, in some sense, that each global phenomenon of A can be isomorphically reproduced by means of B (see Figure 2) and $A \leq_q B$ that each global phenomenon of A can be reproduced by means of B in splitting states of A.

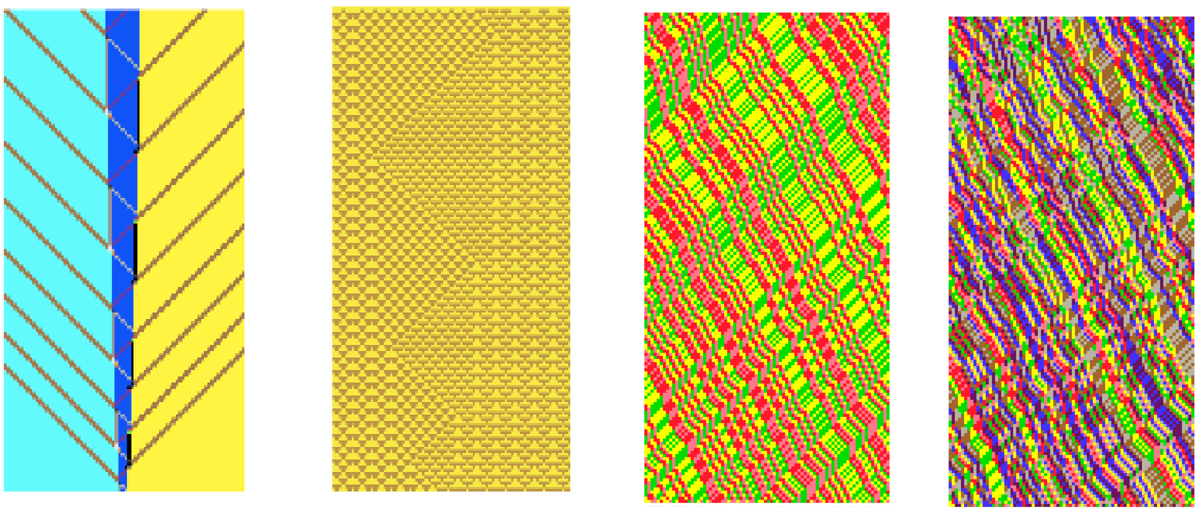

Figure 2: The two left diagrams show two automata A and B. A has 9 states and it is a sub-automaton of B square grouped by 4. The two right diagrams show two automata C and D. C has 2 states and it is a quotient-automaton of D: both states of C are split into 4 states to obtain D but D is not the product of C by another automaton.

Figure 3 represents significant elements of the sub-automaton order structures with square and rectangular groupings. The first important point is that there is a maximum in the case of the rectangular grouping and not in the case of the square one. The second important point is that the maximum contains the set of intrinsic cellular automata of literature [4, 5, 6], i.e. cellular automata which are able to simulate any cellular automaton. Let us observe that if intrinsic universality had already been considered, it is now well formalized in the present algebraic framework; this allows proofs of non-universality but also pertinent comparisons with other notions such as Turing-universality. We also observe that, in both cases, the orders are infinite in width and height, and that there are infinite increasing bounded chains. Understanding the existence of such chains is easy and interesting: if one wants to exhibit some global behavior depending on a parameter n, one needs, for large n, a great amount of states, but this is no more necessary at the upper bound because a new mechanism is introduced which allows to encode the parameter value in the initial configuration.

The same idea can be applied in case of Turing universality with two independent parameters: the number of heads and the ability of a head to make successive zigzags. That allows us to prove that the classes of cellular

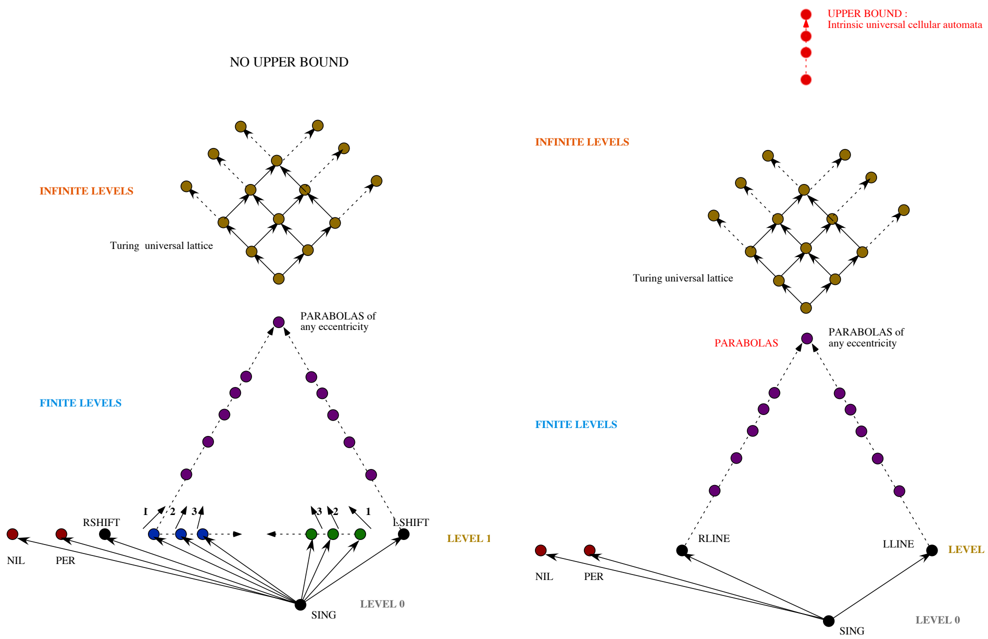

Figure 3: Sub-automaton orders with square and rectangular groupings.

automata simulating Turing universal machines have a structure of lattice (for all sub-automaton orders and orders mixing quotient and sub-automaton). Actually, algebraic hierarchies split Turing universality in an infinite number of classes (inside non trivial order structures) while intrinsic universality is represented in a single class. Moreover, in algebraic classifications corresponding to rectangular and shifted rectangular grouping, the maximum class (of the intrinsic universal cellular automata) is at infinite distance of every other class, especially of these Turing universal classes.

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