Anomalous Stochastic Transport in Complex Systems: microscopic models with anomalous diffusion

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Abstract

It is shown that to describe the "anomalous" stochastic transport - random walks in complex systems the fractional calculus is necessary to apply. The new generalized diffusion equations of fractional order are deduced from microscopic models with anomalous diffusion as Comb model, Continuous time random walks and Levy flights. It is shown that three types of equations are possible : with fractional temporal and fractional spatial derivatives and mixed derivatives. The solutions of these equations are obtained and the physical sense of these fractional equations is discussed. The relation between diffusion and conductivity is studied and the well-known Einstein relation is generalized for the anomalous diffusion case. It is shown that for Levy flight diffusion the Ohm's law is not applied. The new nonlinear response instead Ohm's law current is established. The exponent of nonlinearity is founded, it is connected with the index of anomalous power diffusion.

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1 Introduction.

Classical diffusion, in which diffusing particle hops only to nearest sites, has been thoroughly studied, and many methods, related to the research of this phenomenon, have been developed. In distinction of this the random walks with an anomalous power character are ,however, studied less. One of the well known examples are random walks on percolation clusters (random fractals), which have a sub-diffusion character [1],[2]:

$$< X^2(t) >\sim t^{\frac{2}{2+\theta}} \tag{1}$$

Here t is diffusion time, $< X^2(t) >$ is a random mean square (rms) displacement during the time, θ is a critical index of the anomalous sub- diffusion. Let's note too that the critical exponent of anomalous diffusion θ depends on the space dimension: $\theta_2 \sim 0.8, \theta_3 \sim 1.3$. The change of diffusion character is caused by two reasons : strong tortuous (twistness) of percolation ways and presence of impasses - " dead ends on current ways at least. This problem was formulated many years ago in the [3],[4]- as a problem of "ant in labirint" and it is still not solved.

To take into account an influence of impasses for diffusion character the model of comb stucture was put forward [5] and [6]. This model consists of one-dimensional backbone with fingers of infinite lengths - see fig.1. Using the technique of the generating functions it was shown, that the root-mean-square displacement along an axis of structure depends on time in the anomalous way (1) with the exponent $\theta = 2$.

This model is one of the few exactly solvable models with unusual diffusion properties. So in this paper we consider this model in more detail. The generalized diffusion equation, describing random walks along an axis of structure, was deduced. It essentially differs from the usual diffusion equation, having the form of the continuity equation: instead of the first derivative on time the derivative of the fractional order 1/2 arises. The expression for a diffusion current remains the former - see also [7]. The generalization for a multidimensional case is performed. The relation of the diffusion on the comb model with a problem of continuous time random walks (CTRW) is established [8],[9]. A further development of the model is a study of random walks on the comb structure with random distribution of fingers over lengths [10] , [11]. In particular it was shown for a power law distribution $f(l) \sim l^{-\gamma}, 1 \leq \gamma \leq 2$ rms depends on time in the following power way :

$$< X^2(t) >\sim t^{\frac{\gamma}{2}} \tag{2}$$

It is connected with that at these values $1 \leq \gamma \leq 2$ all moments of the power distribution over lengths are diverged. Recently an analogous behavior was obtained in the continuum description of the anomalous diffusion on the comb structure [12],[13]. It is necessary to note that the diffusion problem on random comb structure is not yet solved. The second part of this paper is devoted to another anomalous random walks - super-diffusion via Levy flights [14],[15]. At Levy flights particles may hop for an arbitrary large distance with a power probability, so that rms displacement per unit time appears to be infinite. The study of Levy diffusion is of interest as a microscopical model with an unusual diffusion and in a connection with possible applications to hopping conductivity in disordered media and to other fields. The generalization of Levy diffusion for a finite length of hop is discussed. In this case Levy flights are alternated by the usual diffusion. The most interesting one is a research on the relation between diffusion and conductivity in the super-diffusion case. It is shown that due to a super-diffusion character of random walks the current and electric field are connected in a nonlinear way. The index of the nonlinearity is described by the exponent of the anomalous Levy diffusion.

The paper organized as follows. In section 2 the exact solution of the random walks on comb structure is obtained. Namely the generalized fractal diffusion equations for the anomalous case are deduced in two different ways. In section 3 the generalization for a multidimensional case is made. The connection between problems of diffusion on comb structure and continuous time random walks is considered in section 4. The drift on the comb structure is considered in section 5. In the last sections 6-8 the diffusion via Levy flights and in an electric field are studied. It is shown that a relation between diffusion and conductivity is nonlinear. Section 9 concludes the paper and the discussion of results is given.

2 The diffusion on comb structure.

A feature of the diffusion in the considered model consists of that the displacement in the X-direction is possible only along an axis of structure (at y = 0). This means that diffusion coefficient D_{xx} is different from zero only at y = 0:

$$D_{xx} = D_1 \delta(y) \tag{3}$$

i.e. X- component of the diffusion current is equal to:

$$J_x = -D_1\delta(y)\frac{\partial\rho}{\partial x} \tag{4}$$

The diffusion along fingers is considered as usual: $D_{yy} = D_2$. Thus, the random walks on the comb structure is described by the tensor of diffusion:

$$\hat{D} = \begin{pmatrix} D_1\delta(y) & 0 \\ 0 & D_2 \end{pmatrix}$$

Accordingly, we obtain the following diffusion equation:

$$[\frac{\partial}{\partial t} - D_1\delta(y)\frac{\partial^2}{\partial x^2} - D_2\frac{\partial^2}{\partial y^2}]G(x,y,t) = \delta(x)\delta(y)\delta(t) \tag{5}$$

Here $G(x,y,t)$ is the Green function of the diffusing problem. To solve the equation we use the following reception. Let's rewrite equation (5) as the usual diffusion equation with a non-uniform right part:

$$[\frac{\partial}{\partial t} - D_2\frac{\partial^2}{\partial y^2}]\rho = D_1\delta(y)\frac{\partial^2\rho}{\partial x^2} \tag{6}$$

The solution of the homogeneous equation (6) is well known and has the Gaussian form:

$$G(y,t) = \frac{exp(-\frac{y^2}{4D_2t})}{\sqrt{\pi D_2 t}} \tag{7}$$

Thus we obtain the integral equation for the concentration of the diffusing particles:

$$\rho(x,y,t) = \int G(y-y', t-t')D_1\delta(y')\frac{\partial^2\rho(x,y',t')}{\partial x^2}dy'dt' \tag{8}$$

After integration over y' one obtains the closed equation for the concentration of particles on an axis of the structure (y=0):

$$\rho(x,o,t) = D_1\frac{\partial^2}{\partial x^2}\int_{-\infty}^{t}\frac{\rho(x,o,t')}{\sqrt{\pi D_2(t-t')}}dt' \tag{9}$$

It is easy to see, that the right-hand side of formula (9) is the integral of the fractional order 1/2 [16],[17]. Therefore using the operator of fractional differentiation of a degree 1/2 we obtain the required diffusion equation:

$$\frac{\partial^{\frac{1}{2}}\rho(x,t)}{\partial t^{\frac{1}{2}}} = D_1\frac{\partial^2\rho(x,t)}{\partial x^2} \tag{10}$$

The integro-differential form of the diffusion equation (10) is a consequence of random disappearance and subsequent birth of particles at the axis of structure at diffusion (leaving and returning to an axis of structure). Let's mark that this equation describes the diffusion problem with a non-conserving number of particles.

To find the solution at arbitrary values of the coordinate y we use another direct approach. Let's use a mixed (s, k,y)-representation :

$$[s + D_1 k^2 \delta(y) - D_2 \frac{\partial^2}{\partial y^2}]\rho(s, k, y) = \delta(y) \tag{11}$$

Let's find the solution (11) in the form :

$$G(s, k, y) = g(s, k)exp(-\lambda|y|) \tag{12}$$

After necessary calculations one has :

$$G(k, y, s) = \frac{exp(-(\sqrt{\frac{s}{D_2}}|y|)}{2\sqrt{sD_2} + D_1 k^2} \tag{13}$$

Using Fourier transformations , we obtain :

$$G(x, y, t) = \int_0^\infty \frac{exp(-\frac{x^2}{4D_1\tau} - \frac{-D_2(\tau+|y|)^2}{4t})}{\sqrt{2\pi t^3}} \tag{14}$$

To obtain this expression the following identity was used:

$$\int_0^\infty exp(-\alpha\tau)d\tau = \frac{1}{\alpha} \tag{15}$$

The distribution of particles on an axis of structure is described by the same expression at y=0.

Let's note that the complete number of particles on an axis of structure decreases or in other words this diffusion problem is the one with a non-conserving number of particles:

$$< G >= \int G(x, o, t)dx = \frac{1}{\sqrt{2D_2 t}} \tag{16}$$

Taking into account last remark to calculate the displacement along an axis of structure:

$$< X^2(t) >= \frac{< X^2 G >}{< G >} = D_1\sqrt{\frac{t}{D_2}} \tag{17}$$

Let's return to the equation for $G(x, 0, t)$. As follows from (13) in (s, k) - representation it has the form:

$$[2\sqrt{sD_2} + D_1 k^2]\rho(s, k) = 0 \tag{18}$$

It is easy to see that this equation consists of the Fourier representation of the fractional derivative on time [16],[17],[18]. So we recover the diffusion equation for a density of particles on an axis of the structure in the form (10).

So the consideration of random walks on comb structure shows that the problem with anomalous diffusion and with non-conserving number of particles should be described by the diffusion equation with temporal derivative of the fractional order.

3 Multidimensional case.

Let's generalize these results for a multidimensional case. First let's begin with a three-dimensional comb structure. Such a structure is formed by attaching the additional fingers to the existing two-dimensional comb structure that points in the direction parallel to the Z axis. Hence in the three-dimensional case displacements in the X-direction are possible only along the intersections of the planes $y = 0$ and $z = 0$. In other words the diffusion coefficient is not zero , i.e. $D_{xx} = D_1\delta(y)\delta(z)$. Accordingly , a displacement in the y-direction is possible only if $z = 0$, and a displacement along z axis is ordinary. Thus, we have the following diffusion tensor:

$$\hat{D} = \begin{pmatrix} D_1\delta(y)\delta(z) & 0 & 0 \\ 0 & D_2\delta(y) & 0 \\ 0 & 0 & D_3 \end{pmatrix}$$

So the corresponding diffusion equation in the mixed (s, k, y, z) - representation is :

$$[s + D_1 k^2\delta(y)\delta(z) - D_2\delta(z)\frac{\partial^2}{\partial x^2} - D_3\frac{\partial^2}{\partial y^2}]\rho(s, k, y, z) = 0 \tag{19}$$

Let's find a solution for (19) in the form:

$$\rho(s, k, y, z) = g(s, k)exp(-\lambda_2|y| - \lambda_3|z|) \tag{20}$$

Substituting (20) into Eq. (19) yields the following formulas for the parameters λ_2 and λ_3 and the function $g(s,k)$:

$$\lambda_3^2 = s/D_3,\ \lambda_2^2 = \frac{2\lambda_3 D_3}{D_2},\ \ g(s,k) = \frac{1}{2\lambda_2 D_2 + D_1 k^2} \tag{21}$$

Consequently for the mean-square displacement along the x and y axes we then have :

$$< X^2(t) >\sim t^{1/4},\ < Y^2(t) >\sim t^{1/2} \tag{22}$$

Hence in the N-dimensional case the diffusion tensor is described by the matrix :

$$\hat{D} = \begin{pmatrix} D_1\delta(x_2)\ldots\delta(x_N) & 0 & \ldots \\ 0 & D_2\delta(x_3)\ldots\delta(x_N) & \ldots \\ \vdots & \vdots & \vdots \\ \ldots & D_{N-1}\delta(x_N) & 0 \\ 0 & \ldots & D_N \end{pmatrix}$$

Accordingly we find a solution for the N-dimensional diffusion problem in the form

$$\rho(s,k,x_2,x_3\ldots,x_N) = g(s,k)exp(-\lambda_2|x_2| - \lambda_3|x_3| - \ldots - \lambda_n|x_N|) \tag{23}$$

Here the parameters λ_N are linked through the formulas:

$$2\lambda_N = s/D_N,\ \lambda_{N-1}^2 = \frac{2\lambda_N D_N}{D_{N-1}}, \ldots,\ \lambda_2^2 = \frac{2\lambda_3 D_3}{D_2} \tag{24}$$

and the function $g(s,k)$ is defined in the expression (21). The formulae (23) and (24) give the complete solution of the multidimensional problem. For instance it is easy to calculate the mean-square displacement along the main axis of the structure:

$$< X_N^2(t) >\sim t^{1/2(N-1)} \tag{25}$$

For the next lateral finger the mean-square displacement is

$$< X_{N-1}^2(t) >\sim t^{1/2(N-2)} \tag{26}$$

...And for the axis, from which only fingers of infinite length emerge, we have

$$< X_2^2(t) >\sim t^{1/2} \tag{27}$$

Thus random walks on a multidimensional comb structure is of a hierarchical nature and there are many variants of behavior of the mean-square displacements along the axes of the structure.

4 Continuous-time random walks.

The above problem of a random walk on an N-dimensional comb structure is connected to the problem of diffusion in a medium with traps (continuous - time random walk). The difference between the two problems consists in that in diffusion in a medium with traps the particles do not disappear, but only delay at each site with a certain probability. The total number of diffusing particles is conserved [19], [20]. For a comb structure the transition to the problem with a continuous distribution over delay time occurs if we study the following quantity:

$$\tilde{G}(x,t) = \int G(x,y,t)dy \tag{28}$$

According to (13) the function $\tilde{G}(x,t)$ is described by the equation :

$$[s + \frac{D_1 k^2 s^{1/2}}{D_2}]\tilde{G} = 1 \tag{29}$$

Hence in the case of a medium with traps the diffusion equation has the form of the continuity equation for a medium with temporal dispersion:

$$\frac{\partial \rho(x,t)}{\partial t} - \frac{\partial J}{\partial x} = 0 \tag{30}$$

where

$$J = -\frac{D_1}{2D_2}\frac{\partial}{\partial x}\int \frac{\partial \rho(x,\tau)}{\partial \tau}\frac{\partial \tau}{|t-\tau|^{1/2}} \tag{31}$$

Diffusion is still anomalous with the exponent $\theta = 2$. Let's consider the three-dimensional case and examine the Green function averaged over the y and z axes , i.e. the function $\tilde{G}(s,k) = \int\int G(s,k,y,z)dydz$. According to (23), for this function, we have the equation:

$$[s + D_1 k^2 (\frac{4sD_3}{D_2})^{3/4}]\tilde{G} = 1 \tag{32}$$

Hence the diffusion equation has the form of the continuity equation with a diffusion current:

$$J \sim -\frac{\partial}{\partial x}\int \frac{\partial \rho(x,\tau)}{\partial \tau}\frac{\partial \tau}{|t-\tau|^{3/4}} \tag{33}$$

Further we study the Green function averaged over one coordinate z :

$$\tilde{G}(s,k,y,t) = \frac{exp(-\lambda_2|y|)}{\lambda_3(2\lambda_2 D_2 + D_1 k^2)} \tag{34}$$

Accordingly, the motion along the axis $y = 0$ is described by the equation:

$$[s^{3/4} + ADk^2(s)^{1/2}]\tilde{G} = 0 \tag{35}$$

where $A = const$.

The number of particles on the $y = 0$ axis is not conserved because particles are also in the dead ends . As result of this the diffusion current contains a fractional temporal derivative of order 1/2. So in the N-dimensional case the equation for the function $\tilde{G}_m$, averaged over the m coordinates has the form:

$$[s^\beta + s^\nu k^2]\tilde{G}_m(s,k) = 0 \tag{36}$$

where $\beta = (N - m + 1)/4$ and $\nu = (N - m - 1)/4$

5 Drift on the comb structure model.

The appearance of the electrical field leads to an anisotropy of random walks. In weak fields the anisotropy parameter $\alpha(E) << 1$ is small and is proportional to a field. Accordingly the field current equals: $J = n\mu E$. In the comb structure the mobility tensor is analogous to the diffusion coefficient. The equation for the diffusion on comb structure and in an electrical field has the following form:

$$[\frac{\partial}{\partial t} - \delta(y)(D_1\frac{\partial^2\rho}{\partial x^2} + \mu_1\frac{\partial^2}{\partial y^2}) - (D_2\frac{\partial^2}{\partial y^2}\mu_2\frac{\partial^2}{\partial y^2})]\rho(x,y,t;E) = 0 \tag{37}$$

Let's assume that the field is directed only along an axis of structure $\vec{E} = E(1,0,0)$. Accordingly,the Green function in mixed (s,k,y) -representation is equal to:

$$G(s,k,y;E) = \frac{exp(-\sqrt{s/D_2}|y|)}{2\sqrt{sD_2} + D_1k^2 + ik\mu_1 E} \tag{38}$$

After Fourier transformations , we obtain :

$$G(x,y,t) = \int_0^\infty \frac{exp(-\frac{(x-\mu_1 E\tau)^2}{4D_1\tau} - \frac{-D_2(\tau+|y|)^2}{4t})}{\sqrt{2\pi t^3}} \tag{39}$$

Let's find the first moment of the Green function in a field:

$$< X(t) >= \mu_1 E\sqrt{\pi t/2} \tag{40}$$

Let's emphasize that the response to a constant electrical field appears as a time-dependent one. Namely, the velocity decreases with time according to the power way:

$$< V >= \mu_1 E\sqrt{\pi/2t} \tag{41}$$

This result means that in the anomalous diffusion problem with drift it is impossible to find such an inertial system of the coordinates, which is moved with constant speed and in which the diffusion remained only as in the usual diffusion case.

Let's consider also the influence of an electrical field on a returning probability. In the usual diffusion case the drift leads to the exponential reduction of it:

$$G(o,t;E) = \frac{exp(-(\mu E)^2 t)}{\sqrt{\pi t}} \tag{42}$$

In our case it is easy to see that for large time values there is only power reduction of the probability:

$$G(o,t;E) \sim ((\mu E)^2 t)^{3/4} \tag{43}$$

This result can be easily understood. The electrical field acts on particles only when they are on a structure axis. But most of the time a particle remains on the fingers, outside the axis, so a more slightly power dependence is obtained.

6 Levy flight diffusion.

As it was discussed above another microscopical model with anomalous diffusion is a model with Levy flight diffusion. A feature of the Levy flight diffusion is that in each step a particle may move for an arbitrarily large distance, so that the root-mean-square displacement per unit time appears to be infinite [14]. Numerical simulation of diffusion via Levy hops shows that the points visited by a diffusing particle form spatially well-separated clusters. From more in-depth consideration one can see that each cluster consists of a set of clusters, so that a structure of self-similar clusters appears [15]. So one can say that Levy diffusion is a random walk among self-similar clusters.

The probability distribution function in the Fourier representation has the form:

$$P(k,t) \propto e^{-A|k|^{\mu} t} \tag{44}$$

where A and μ are positive magnitudes, $1 < \mu < 2$. Such stable distributions are called Levy distributions. A more detailed discussion of Levy hops is given in [21].

The study of Levy diffusion is of interest as a microscopical model with unusual diffusion, but also in connection with some possible applications, for example , to the hopping conductivity problem in inhomogeneous medium [22].

6.1 Discrete distribution of Levy random walks.

Let us consider a one-dimensional discrete analog of a Levy flight [14]. Let the probability, that a particle occupies the l-th site after n steps, be $P_n(l)$ and let $f(l)$ be the probability distributions of hops over lengths. So the master equation for complex diffusion has the form:

$$P_{n+1}(l) = \sum_{m=-\infty}^{\infty} f(l-m) P_n(m) \tag{45}$$

To simulate a Levy flight the following function is used for $f(l)$:

$$f(l) = \sum_{n=0}^{\infty} a^{-n} (\delta_{l,-b^n} + \delta_{l,b^n}) \tag{46}$$

where $\delta_{n,m}$ is the Kronecker delta and a and b are the parameters of the Levy flight. Then after Fourier transformation the structure function for such a random walk is equal to:

$$\lambda = \int f(l) exp(ikl) dl = \sum_{n=0}^{\infty} a^{-n} \cos(kb^n) \tag{47}$$

Note that the structure function $\lambda(k)$ satisfies the functional equation:

$$\lambda(k) = a\lambda(kb) + cos(k) \tag{48}$$

Hence at $k \to 0$ the structure function is a power law function with exponent $\mu = ln(a)/ln(b)$. One can establish the non-analytic power-law behavior at $k \to 0$ by means of a Mellin transformation, or with the help of Poisson formulae for set summation . For details see [14].

6.2 Transition from ordinary diffusion to Levy diffusion.

In this section , in addition to Levy hops we allow for ordinary diffusion. The simplest way to do this is to introduce a finite hop length ξ at each step. So we obtain a random walk in which ordinary diffusion alternates with Levy hops. However, due to the super-linear time dependence of the rms displacement for Levy diffusion, on small scales (times) the main contribution to the random walk is provided by ordinary diffusion, while at long times Levy hops contribute most to the random walks. Accordingly, the hop-length distribution function has the form:

$$f(l) = \sum_{n=0}^{\infty} a^{-n}(\delta_{l,-(b^n+\xi)} + \delta_{l,(b^n+\xi)} \tag{49}$$

Hence the structure function is :

$$\lambda = \sum_{n=0}^{\infty} a^{-n} \cos(kb^n + k\xi) \tag{50}$$

In the limit of small length ($b \to 0$) this formula turns into the expression corresponding to ordinary diffusion:

$$\lim_{b \to 0} \lambda(k,\zeta) = \frac{a-1}{a} cos(k\xi) \tag{51}$$

7 Nonlinear relation between diffusion and conductivity.

7.1 Einstein relation and its generalization.

Below the particle drift or the relation between diffusion and conductivity is studied when there is Levy diffusion in the system. For the case of usual classical diffusion and linear response (Ohm's law) this problem was considered by A. Einstein and the well-known Einstein relation was obtained. However in the case of Levy hops a question about the existence of an Einstein relation arises. The problem is that the diffusion coefficient, defined in the usual way as $D = \lim\limits_{t \to \infty} \frac{x^2(t)}{t}$, diverges in a Levy flight diffusion case.

Consequently, there are two possibilities: Either the particle mobility tends to infinity, which is nonsense from a physical point of view, or the

Einstein relation is broken. Below it will be shown that instead of the Einstein relation a new nonlinear relation between mobility and diffusion coefficient appears.

Let us recall the well-known Einstein arguments. Let there be in the system the diffusion $J_d = -D\nabla n$ and the field $J_f = \mu E n$ currents. In the equilibrium the diffusion current J_d is compensated by the field current J_f, and the distribution function must have Boltzmann's form:

$$J_d + J_f = 0,\ N_{eq} \propto e^{-U/kT} \tag{52}$$

where U is the potential energy, T is temperature, and k is Boltzmann's constant.

Before applying analogous arguments to Levy flights consider the assumptions used in deriving the Einstein relation. There are the three following assumptions:

i) the Boltzmann's statistics

ii) the expression for the diffusion current in the usual classical form

iii) the linear Ohm's law

Let us try to understand which of these assumptions need to be modified. Firstly, the assumption about Boltzmann's statistics is not essential, since its type is determined by the statistical properties of the system, and we will retain it. Secondly, the diffusion current has a different form and we write it in a general operator form:

$$J_d = -\hat{K}n = -iA\vec{k}|k|^{\mu-2}n \tag{53}$$

And finally we write the field current as $J_f = nV$, where V is the drift velocity.

By taking a definition for the derivative of the fractional order in the form of the set [23], one can get a general formula for the drift velocity:

$$\vec{V} = e^{U/kT} \lim_{\epsilon \to 0} (\triangle^2 + \epsilon)^{(\mu-2)/4} \nabla \exp(-\frac{U}{kT}) \tag{54}$$

where $\triangle$ is the Laplace operator.

In a homogeneous electrical field $U = -qEr$ we recover that the drift velocity depends on the electric field in a nonlinear way:

$$V = Aq\vec{E}\frac{|q\vec{E}|^{\mu-2}}{(kT)^{\mu-1}} \tag{55}$$

It should be emphasized that this nonlinearity occurs in arbitrarily weak fields and is a consequence of the unusual character of diffusion. The power of nonlinearity is described by the critical index of the Levy hop diffusion.

This is a preliminary result, which we obtain below in an exact way.

8 Random walks of Levy and particle drift in the electric field.

Let us now introduce an anisotropy into the random walk on self-similar clusters. By virtue of the specific nature of Levy hops a particle can move in one hop over an arbitrary distance b^n. For this reason a small anisotropy $(1 + \alpha)$, with $\alpha = qEs/kT$, when particles move on a small distance s, becomes exponentially large on large distances b^n. Since at each step a diffusing particle leaves a site, the sum of probabilities W_+ and W_- of motions parallel and anti-parallel, respectively, to the field must be equal to 1:

$$W_+ + W_- = 1$$

Hence we get the expressions for probabilities of motion parallel and anti-parallel to the field:

$$W_\pm = \frac{(1 \pm \alpha)^{b^n}}{(1+\alpha)^{b^n} + (1-\alpha)^{b^n}} \tag{56}$$

Therefore, the structure function $\lambda(k; E)$ in the case of diffusion via Levy hops in the electrical field equals:

$$2\lambda(k; E) = \sum_{n=0}^{\infty} a^{-n}[\cos(kb^n) + i\sin(kb^n)(W_+ - W_-)] \tag{57}$$

As for usual diffusion the second term contains the drift velocity for small $k \to 0$:

$$V = i\frac{\partial\lambda(k; E)}{\partial k} \mid_{k\to 0} = \sum_{n=0}^{\infty}(\frac{b}{a})^n * (W_+ + W_-) \approx \sum_{n=0}^{\infty}(\frac{b}{a})^n \tanh(\alpha b^n) \tag{58}$$

where $\tanh(x)$ is the hyperbolic tangent.

Using the Poisson formula we obtain after some calculations the formula for the velocity:

$$V(E)=\alpha/2+\alpha^{\mu-1}\left[\sum_{m=-\infty}^{\infty}\int_{1}^{\infty}\tanh(z)z^{-\gamma_m}dz+\int_{o}^{\alpha}\tanh(z)z^{-\gamma_m}dz\right] \quad (59)$$

where the exponent is equal to:

$$\gamma_m=\mu+2\pi im/\ln b.$$

It is easy to see that for weak fields the second term in brackets is less than the first term. Thus, in arbitrarily weak electric fields one can get the nonlinear field dependence of velocity (55).

8.1 Transitions from ordinary diffusion to Levy diffusion and from Ohm's law to nonlinear response.

Anisotropy is introduced into these random walks using the method described above: we replace the hop length with the quantity $b^n+\xi$. Thus the structure function in an electric field and for finite hop length is:

$$\lambda(k,\xi,\alpha)=\sum_{n=0}^{\infty}a^{-n}[\cos(kb^n+k\xi)+i\sin(kb^n+\xi)(W_+-W_-)] \quad (60)$$

And after calculations by Poisson's method we obtain the following results: in arbitrarily weak fields the velocity is nonlinear in the field, eq. (55), and crosses over to linear behavior in strong fields:

$$V\simeq E\xi^{2-\mu}, \qquad qE\xi/kT\gg 1 \quad (61)$$

Thus the particle velocity in an electric field has two asymptotic limits in accordance with two diffusing regimes: Levy hops and ordinary diffusion.

9 Discussion.

We have studied random walks on the comb model and found that the existence of fingers on comb model - analog of "dead ends" in the current-carrying paths of percolation systems leads to the anomalous nature of the random walk. We have established that for diffusion problems , in which the number of particles is not conserved, the generalized diffusion equations must be the fractal temporal derivative equations : instead of a first temporal derivative , the equation must contain the fractional-order derivative. Fractional temporal derivatives emerge due to the random disappearance and reappearance of diffusing particles (the departure of particles from axis and their return). Let's stress that in our consideration the fractal temporal diffusion equations are deduced in an obvious way. The physical sense of fractional temporal derivative is clear. Usually the fractal diffusion equations are postulated [25], [26] and [18], and questions about the possibility of its application are arised.

When we examine random walks in a medium with traps, the same problems appear. As noted earlier, the problem of diffusion in a medium with traps differs from the problem of diffusion along the axis of a comb structure. The difference lies in the fact that the particles do not disappear, but delay at each site with a certain probability. The total number of diffusing particles is conserved. In other words we have the law of mass conservation, expressed by a continuity equation. However, the anomalous nature of diffusion, due to the capture of particles by the traps, leads to an unusual expression for the diffusion current with fractional temporal derivative. Note that mathematically the generalized diffusion equations in both problems are different and describe different physical situations. First, in a diffusion along the axis of a comb structure the number of particles is not conserved. Second, the diffusion currents are different.

The generalized relation between diffusion and conductivity is obtained for a sub-diffusion case. It has the form of the well- known Einstein relation for the diffusion coefficient and the particle mobility, depending on the time.

In the second part of the paper the Levy flight diffusion is considered. The main result consists of the nonlinear dependence of the particle mobility in weak electric fields. Usually theoreticians expand the current in powers of the electric field of the electric field:

$$J = \sigma E + \chi |E|^2 E + \dots \tag{62}$$

Our result essentially differs from those , obtained by such a method. In the microscopical model of Levy hops we show that current depends on an electric field in a nonlinear way due to unusual regime of diffusion in space, i.e. there is no linear term, corresponding to Ohm's law, in the field expansion of the current (62). In other words if there is an usual diffusion in the system, so the Ohm's law exists , in the case of anomalous diffusion as Levy hops the response of system has a nonlinear character.

We consider the transition from ordinary diffusion to Levy flight by introducing a finite displacement length ξ at each step. The new parameter $qE\xi/kT$, which determines whether the particle mobility behaves linearly or nonlinearly, appears in the problem. In other words a new physical length L_E governed by the electric field emerges in such diffusion problems:

$$L_E = \frac{kT}{qE} \tag{63}$$

To appreciate the significance of this quantity we consider an ordinary random walk in an external electric field. Let's imagine that the medium is partitioned into the blocks of size L_E. Then we study the particle behavior within a single block. With a probability of order unity the particles leave the block when it moves along the field and does not leave the block when it moves against the field. Briefly speaking within a block, whose linear size is of order L_E, ordered motion prevails over diffusion. This makes it possible to estimate the particle velocity to be:

$$V = \frac{L_E}{t_E} \tag{64}$$

where t_E is the diffusion time for the distance L_E. For ordinary diffusion $t_E = L_E^2/D$ and we have the well-known Einstein relation:

$$V = q^2 DE/kT \tag{65}$$

For a Levy flight diffusion the same estimates give the nonlinear dependence of velocity. And for the case of two diffusion limits we have two different: linear and nonlinear expressions for mobility. Recently the deduced nonlinear behavior of the velocity due to the unusual nature of diffusion was confirmed by the independent numerical simulations of particle drift in the presence of Levy diffusion [24].

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