Noise Sensitivity of Portfolio Selection under Various Risk Measures

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Extended Abstract

The theory of portfolios, initiated by Markowitz, has suffered from the "curse of dimensions" from the very outset. Whereas the computational difficulties associated with the selection of the optimal portfolio have been greatly alleviated by the progress of information technology, the fundamental problem of insufficient input data and the resulting estimation error remain serious stumbling blocks. Even if we disregard the notoriously difficult problem of estimating returns and focus exclusively on the minimal risk portfolio, the amount of information contained in the available finite-length time series is typically far below the amount of information necessary for the reliable determination of the optimal portfolio. As for a portfolio of size N and time series of length T, the number of input data is NT, whereas the number of data needed for the construction of the covariance matrix is $O(N^2)$, we expect that the quality of the estimate essentially depends on the ratio N/T and that the error goes to zero only in the limit of very small N/T. Now the problem is that for typical bank portfolios N/T is never sufficiently small, in fact, it may well be larger than unity, the threshold value where the covariance matrix becomes singular and the portfolio selection problem meaningless.

Over the past decades a large number of different techniques have been developed to tackle this problem and reduce the effective dimension of large bank portfolios, but the efficiency and reliability of these procedures are hard to assess or compare. In this paper we propose a model (simulation)-based approach which can be used for the systematic testing of all these dimensionality-reduction (filtering) techniques [1]. To illustrate the usefulness of our framework, we develop several toy models (including a single-index or market model and a market plus sectors model) that display some of the main characteristic features of empirical correlations and generate artificial time series from them. Then, we regard these time series as empirical data and reconstruct the corresponding correlation matrices which will inevitably contain a certain amount of noise, due to the finite length of the time series. For multivariate normal portfolios and asymptotically large N and T with N/T fixed we derive a simple analytic formula for the relative error in the portfolio.

Now we apply several correlation matrix estimators and dimensionality-reduction techniques introduced in the literature and/or applied in practice. As in our artificial world the only source of error is the finite length of the time series and, in addition, the "true" model, hence also the "true" correlation matrix, are precisely known, we can meaningfully compare the performance of the various noise-reduction techniques. One of our recurrent observations [2] is that the recently introduced filtering technique based on random matrix theory (RMT) [3] performs consistently well in all the investigated cases. Based on this experience, we believe that our simulation-based approach can also be useful for the systematic investigation of several related problems of current interest in finance.

In addition to correlated Gaussian returns, we also consider non-stationary time series of the IGARCH(1,1) type, closely related to the exponentially weighted moving average technique implemented in RiskMetrics [4]. In order to be able to apply the RMT-based filtering technique in this context, we have derived the spectrum of a random covariance matrix where the returns are

exponentially weighted with time [5]. Applying this method to *empirical* data we find that the effect of risk depends on the weight factor, whose optimal value corresponds to the trade off between discarding too many past data, thereby destroying the statistics, or retaining too many, hence including non-stationary effects. We determine the optimal weight factor and find that it is considerably larger than the value advocated in RiskMetrics.

As a further attempt to go beyond the classical mean-variance framework, we have also studied the effect of noise on portfolio selection under some alternative risk measures. In particular, we have studied the case of mean absolute deviation (MAD), as described in ref. [6]. The level surfaces of risk under MAD are polyhedrons (instead of the ellipsoidal iso-risk surfaces corresponding to variance), and this leads to an increased sensitivity to noise. We observe a similar effect under the use of expected shortfall (ES) or conditional value at risk which is strongly promoted in the academic literature as the simplest of the coherent risk measures [7]. In addition, portfolio optimization under this measure has been shown to be reducible to linear programming [8] which might, in principle, allow one to optimize extremely large portfolios at a relatively light computational cost. As we show here, the downside is a strongly increased sensitivity to noise. One might think that this is due to the fact that ES, as a kind of conditional expectation, omits a large amount of input data by concentrating only on those above a (typically high) confidence level. A systematic study of the problem reveals, however, that the enhanced sensitivity remains true even for as low confidence levels as 60% or 50%, where ES can be compared with semi-variance. The fundamental reason of this high noise-sensitivity of ES is not understood at present. At the other extreme, for confidence levels approaching 100%, we have a risk measure that can be called worst loss (WL). Although over-pessimistic, this still has the virtue of coherence. Not surprisingly, WL is found to be very sensitive to noise again.

In the course of our studies of the noise-sensitivity of the risk measures ES and WL we have observed a striking phenomenon. As we have already mentioned, the portfolio selection problem within the mean-variance framework becomes meaningless for $N/T > 1$. The same is true for all the other risk measures studied in this paper. On the other hand, for $N/T < 1$, optimization under the variance and also under MAD always has a solution, even if it may be strongly influenced by noise for N/T not small enough. In contrast to this, the optimization under ES and WL does not necessarily have a solution even under the threshold $N/T = 1$, instead, the existence of a solution becomes a probabilistic issue: it depends on the sample. We have studied this remarkable phenomenon both analytically and numerically. In the case of Gaussian-distributed (or, more generally, elliptically distributed) assets and under the WL risk measure we have been able to derive a closed formula for the probability of the optimization problem to have a solution and found that this probability goes to unity only for N/T going to zero. Similar behaviour is observed in numerical simulations for the ES measure for the entire range of confidence levels we have studied. This puzzling phenomenon is absent if short selling is excluded, which may be the reason why it had not been observed by previous authors.

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