Partitioning networks into classes of mutually isolated nodes ⋆

J. Díaz[1], A.C. Kaporis[2], L.M. Kirousis[2,3], and X. Pérez[1]

[1] Universitat Politècnica de Catalunya, Departament de Llenguatges i Sistemes Informàtics, Campus Nord – Ed. Omega, 240, Jordi Girona Salgado, 1–3, E-08034 Barcelona, Catalunya, {diaz, xperez}@lsi.upc.edu

[2] University of Patras, Department of Computer Engineering and Informatics, GR-265 04 Patras, Greece, {kaporis, kirousis}@ceid.upatras.gr

[3] Research Academic Computer Technology Institute, P.O. Box 1122, GR-261 10 Patras, Greece

Abstract. Following the work of Newman, we model complex networks with random graphs of a given degree distribution. In particular we study the case where all vertices have exactly the same degree (random regular graphs). We survey some recent results on the problem of partitioning such a graph into the smallest possible number of classes of mutually isolated vertices, known in graph theory as the colorability problem. We also describe the solution of an open problem about 5-regular graphs.

1 Introduction

It is not an exaggeration to state that Random Graph Theory is undergoing a revolutionary change. Graph theorists have definitely been expelled from the paradise that Erdős and Rényi [6, 7] had created for them. According to this classical theory, a random graph is the product of repeated Bernoulli trials: for each pair of vertices, decide with a given probability, if they will be connected by an edge or not. Classical random graphs however, despite the deep and complicated mathematics devised for their study, are not of much interest from the complex systems point of view. Local decisions that affect the global picture, emergent properties, self-organization, robustness, resilience, to name only a few issues of interest to complexity science, do not pertain to classical graph theory.

As is well known by now, this state of affairs radically changed just before the turn of the century, when researchers tried to study from a graph-theoretical point of view existing complex systems, live or artificial. The pioneering work of e.g., Watts and Strogatz [12, 13], and Barabási and Albert [4] gave a new twist to classical graph theory. Random graphs according to this new paradigm have properties, like power law distribution of the degrees, or freedom from scale, that reflect the interaction patterns of many complex systems.

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One approach to such models was the one undertaken by Newman et al. [10]. In this approach, instead of modelling a random graph with an edge generating process with a specified attachment rule at each step (generative models), one assumes instead that the graph is uniformly random conditional on its degree sequence. Certainly, this static approach is more restricted in comparison to the generative approach. However, it is often more suitable for the analysis of graph properties like clustering, distribution of component sizes, resilience, thresholds etc.

In this paper, we restrict our study to an important subclass of random graphs with a given degree sequence: that of random regular graphs, ie graphs where all vertices have the same degree. A comprehensive account of the pioneering work on random regular graphs can be found in the paper by Wormald [14]. The present paper concerns a much studied graph parameter, that of the chromatic number of a graph. The chromatic number is defined to be the smallest number of colors needed to color the vertices of a graph, given that no two adjacent vertices get the same color. In network terminology, the chromatic number is the smallest number of classes that one can partition the nodes of a network, so that each class comprises only of mutually isolated nodes.

Molloy [9] proved that 6-regular graphs have chromatic number at least 4, asymptotically almost surely (a.a.s.) with respect to the number of their vertices n. Achlioptas and Moore [1] proved that 4-regular graphs have chromatic number 3 with constant probability. Subsequently, Achlioptas and Moore [2] showed that a.a.s., the chromatic number of a d-regular graph ($d \geq 3$) is k or $k+1$ or $k+2$, where k is the smallest integer such that $d < 2k \log k$. Shi and Wormald [11] showed that a.a.s. the chromatic number of a 4-regular graph is 3, that a.a.s. the chromatic number of a 6-regular graph is 4 and that a.a.s. the chromatic number of a 5-regular graph is either 3 or 4. They also showed that a.a.s. the chromatic number of a d-regular graph, for all other d up to 10, is restricted to a range of two integers. Their proofs were algorithmic.

The above results leave open the question of whether the chromatic number of a 5-regular graph can take the value 3 with constant probability (or perhaps even a.a.s.). The difficulty of devising a rigorously analyzable algorithm that provides a 3-coloring for a 5-regular graph with constant probability was explained by research in physics. Krząkała et al. [8] provided a non-rigorously analyzable algorithm that a.a.s. finds a 3-coloring for a 5-regular graph. However, they also showed that the space of assignments of three colors to the vertices (legal or not, i.e. with no two adjacent vertices with the same color or not) consists of clusters of *legal* color assignments inside which one can move from point to point by steps of small Hamming distance. Additionally they showed that to go from one cluster to another by such small steps, it is necessary to go through assignments of colors that grossly violate the requirement of legality (high-energy color assignments). Moreover, the number of clusters that contain points with energy that is a local, but not global, minimum is exponentially large. As a result, local search algorithms are easily trapped into such local minima (metastable states). Non-local search algorithms however are usually not amenable to rigorous analysis.

The above considerations left as the only plausible alternative to try to prove that 5-regular graphs are 3-colorable with constant probability in an analytic way. A technique that has been used towards similar ends is the Second Moment Method. The basic ingredient of this method is the fact that if X is a non-negative random variable (r.v.) then the probability that X is positive is bounded from below from the ratio of the square of its first moment to its second moment. We call this ratio the *Moment Ratio*. Symbolically:

$$\mathbf{Pr}\left[X > 0\right] \geq \frac{(\mathbf{E}\left[X\right])^2}{\mathbf{E}\left[X^2\right]}. \tag{1}$$

This technique was used [3] to solve the long-standing open problem of computing the two possible values of a random graph. In this work the authors considered as X the r.v. that gives the number of balanced 3-colorings of a graph (balanced are the colorings where there is an equal number of vertices of each color). The same r.v. however proved not to work for the case of 5-regular graphs.

In a recent work [5], the r.v. that counts *stable* balanced 3-colorings was considered. These are balanced 3-colorings with the property that for no single vertex v can one change its color without the appearance of an edge with the same color at its endpoints.

It turned out that for stable balanced 3-colorings the variance diminishes (as expected) but the square of the expectation diminishes in a lesser degree and as a result the Moment Ratio becomes asymptotically positive. Thus a non-empirical proof that 5-regular graphs are 3-colorable with positive probability was obtained.

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