On stability of computations by cellular automata

Bruno Durand* and Andrei Romashchenko**

Key words: cellular automata, reliable computations, Toom's rule

Abstract. In this paper, we study stability of computations in the presence of random faults (noise). We focus on homogeneous models such as cellular automata. We present a new proof of stability of Toom's 2-dimensional automaton. The arguments are based on the methods from the famous "Positive Rates" paper by P. Gács. The advantage of our construction is that it explains precisely how errors spread in the computational array and how they are stabilized. Also we show that the same technique can be used to prove correctness of a 3-dimensional fault tolerant computational array.

1 Introduction

In most models of computation, programs are not stable towards faults: if a single step is corrupted, then the result of the computation is erroneous. A task of theoretical and practical interest is to implement reliable computations on faulty devices: the problem is to construct models which provide reliable computations even if some elementary step are faulty. Different approaches were proposed, mainly several variants of faulty circuits and cellular automata – see a survey in [1]. In the present paper we focus on computations based on faulty cellular automata.

The first step towards implementation of stable computations on faulty cellular automata, is the following basic problem: construct a cellular automaton that has at least 2 stable configurations. Here we say that a configuration is stable, if with high probability most cells of the array keep the initial state (corresponding to this configuration) for a long (or even infinite!) time despite random perturbation. Note that this problem is interesting for its significance in physics and thus many authors motivate this kind of work by study of phase transition instead of computation.

* Laboratoire d'Informatique Fondamentale de Marseille, CNRS, Université de Provence, 39, rue Joliot-Curie, F-13453 Marseille Cedex 13, France fax: +33 4 91 11 36 02 `Bruno.Durand@lif.univ-mrs.fr`

** Institute for Information Transmission Problems, RAS, 19 Bolshoy Karetny, 101447, Moscow, Russia, fax: +7 095 209 0579 `anromash@mccme.ru` The work was supported in part by Russian Science Support Foundation and RFBR grants 03-01-00475, 358.2003.1.

The first result in this area belongs to Andrei Toom. In [2] he presented a very simple two-dimensional stable cellular automaton. Despite apparent simplicity of Toom's cellular automaton (which has only two states) the original proof of its stability was very technical. Several other proofs of Toom's result were then given [3–6], but they all are quite non-trivial. One of the goals of this paper is to provide a simple proof that explains precisely what happens with faults and their consequences in the array of Toom's automata.

Toom's construction was used later in [4] in order to simulate 1-dimensional array of cellular automata by a 3-dimensional array of *stable* automata in real time ("in real time" means that one step of computation of the 1-dimensional *simulated* cellular automata is simulated by one step of the 3-dimensional *simulating* cellular automata). Later, Peter Gács improved this result and implemented fault tolerant computations on 2-dimensional [7] and 1-dimensional [8] arrays of faulty cellular automata. To achieve these results Gács developed interesting technique, which might be useful in other situations. However, the construction by Gács is extremely difficult and long, and is really understood by a very small number of people. This is a serious obstacle to further progress in this field: the problem is fundamental and the construction should be used for many related problems but it cannot be as it is. The author himself notes in [1, 8] that one of the most important open problems in this area is to simplify the proof of his results, i.e, to find an understandable construction of self-correcting computations based on faulty cellular automata.

In this paper we deal with the first (and the easiest) part of this construction. Using ideas from [8], we present a new simplified proof of stability of the Toom 2-dimensional automaton, and also correctness of the three-dimensional reliable cellular array proposed in [4]. Our main intention is to split the proof into a few independent steps and provide a simple proof for each of them. Our goal is to explain what happens with faults and their consequences: how long 'islands' of disturbed cells can survive and how they shrink.

Our paper is organized as follows: we first discuss in all detail the most simple case: Toom's rule for an infinite 2-dimensional cellular automaton. In Section 3 we explain what happens on finite arrays of faulty automata (torus). In Section 4 we use our construction to explain the embedding of computations into 3-dimensional fault tolerant cellular automaton.

2 Toom's rule on the infinite plane.

We start with a 2-dimensional cellular automaton. Let us consider a 2-dimensional array of cells, the infinite plane $\mathbb{Z} \times \mathbb{Z}$. There is a fixed (finite) automaton in each cell of the space. This automaton has two states: 'alive' and 'dead'. We identify the state of the cell with the state of the automaton it contains. All cells update synchronously as follows: the state of any cell at time $t+1$ is defined as majority of states of itself, its North and its East neighbors at time t, see Fig. 1. The majority is well defined since there are 2 state possibilities for

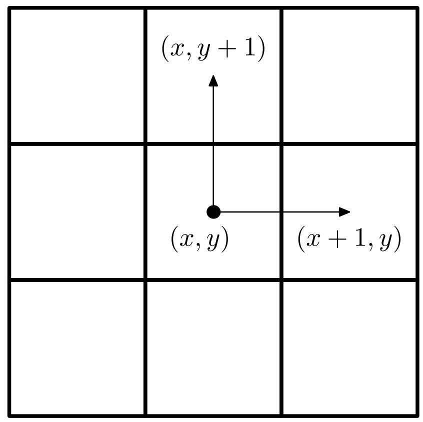

Fig. 1. Toom's rule

each of 3 cells. We call this automaton rule *Toom's rule* since it is inspired by Toom's work [2].

Note that if we have a finite 'island' of n alive points on the infinite plane of dead points, Toom's rule will shrink this island in time at most n. To see this, we circumscribe a triangle that we call 'Toom's triangle' around the island, see Fig. 2. This (virtual) triangle has the property to contain all alive cells, be isocele and minimal.

In course of time, Tooms's rule transforms the island of alive cells. At each step we circumscribe around the island new Toom's triangle. How does the island vary on time, and what happens with the Toom triangle? Obviously, the vertical leg of the triangle may not move to the left, and the horizontal leg triangle may not move downwards. But the hypotenuse moves to the south-west (each step it must shift at least by 1). Thus the triangle shrinks to nothing in a finite number of steps. Moreover, the number of steps required to eliminate the island is not greater than the initial size of Toom's triangle, which is not greater than the number of alive cells in the island.

The arguments above are quite trivial. But we stress that the idea of shrinking of Toom's triangle is the main issue of our arguments. In the sequel we shall

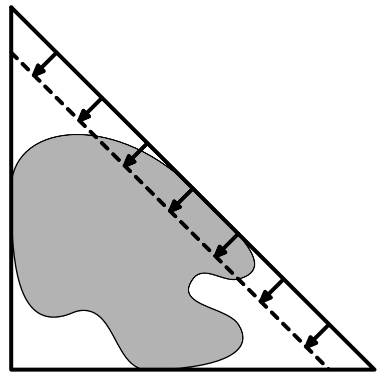

Fig. 2. Toom's triangle

generalize these arguments for a non-trivial case, when the evolution of Toom's triangle becomes more complicated due to random faults.

Now, let us introduce faults. We model them by a random transformation. At any time step, any dead cell might (randomly) come to live. We call such events 'miracles'. Our probabilistic model is as follows: we assume that miracles at different cells and times are independent and that the probability to get a miracle at point a at time t is some (small) $\epsilon > 0$. Beware that miracles are not symmetric: they consist of spontaneous life but not of spontaneous death. This non-symmetry strengthen our result because in order to prove that a dead configuration is stable, we cannot use in our model the spontaneous death of alive cells.

Initially, i.e., at time $t = 0$, all cells are dead.

What is to be proved looks very simple: we want to prove that if ϵ is small enough, then any cell at any time is dead with high probability. Our ideas come from reading of [1, 8, 9].

An Informal plan of our proof is:

1. Separate all miracles into 'islands' of different levels. Islands are sets of miracles in space $\times$ time. An n-level island of miracles should be of size about $q_n \times q_n$ in space and about q_n in time (we specify the values q_n later); an n-level island should contain enough miracles (at least 2^n). Islands of the same rank should be separated from each other in space and time, so than they cannot interfere with each other. The first step of our proof is the fact that such ranking of the set of miracles is possible with probability 1.

2. Show that an n-level island of miracles disappears without consequences in time $\mathcal{O}(q_n)$, without interacting with other islands of the same rank. If there were no lower rank errors, the proof would be trivial: we could circumscribe Toom's triangle around a given island and note that it shrinks as when there are no miracles (the vertical and the horizontal legs may not move left and down respectively, the hypotenuse moves South-West at speed at least 1). In the presence of lower rank errors, the vertical and the horizontal sides *may* move left and down respectively. What we prove is that this movement is sufficiently slow. The speed of the hypotenuse may also be a bit less than 1, but we prove that it remains high enough. This is how we obtain that the triangle vanishes in time $\mathcal{O}(q_n)$.

2.1 Notations and definitions

We denote by $Space$ the set $\mathbb{Z} \times \mathbb{Z}$, and by $Time$ the set $\mathbb{Z}_+$.

Define *a random perturbation* as $\xi(a)$, $a \in Space \times Time$. Let $Prob[\xi(a) = 1] = \epsilon$ and $Prob[\xi(a) = 0] = 1 - \epsilon$. We assume that $\xi(a)$ and $\xi(b)$ are independent for $a \neq b$. Intuitively, when $\xi(a) = 1$ there is "noise" and a miracle may occur in the cell and time addressed by a.

Now we are ready to define Toom's cellular automaton in a faulty situation. For any point $(x, y, t) \in Space \times Time$ the state of a cell $s(x, y, t)$ is 1 (alive) or 0 (dead).

1. $s(x, y, 0) = 0$ for any x, y, i.e., initially all cells are dead;
2. for any $t > 0$ the following transition rule is used: let $a = (x, y, t)$, $a_0 = (x, y, t-1)$, $a_e = (x+1, y, t-1)$, $a_n = (x, y+1, t-1)$; then

$$s(a) = \max\{s'(a), \xi(x, y, t)\},$$

   where $s'(a) = majority\{s(a_0), s(a_e), s(a_n)\}$. In other words, at a noisy point (i.e., if $\xi(a) = 1$) a cell is made alive, independently of its neighborhood. In a normal point (i.e., in a point a such that $\xi(a) = 0$) Toom's rule is applied.

A point $a \in Space \times Time$ is called *a miracle,* if Toom's rule is not applied at this point, i.e., $s(a) \neq s'(a)$. Beware that if the point is noisy and becomes alive, it is not sure that it is a miracle. It could be alive due to Toom's rule, and in this case, even if it is noisy, it is not called a miracle.

We denote by $M \subset Space \times Time$ the set of all miracles. The main idea of the proof is to split M into 'islands' of miracles of different size.

We use l_∞-norm to measure the distance between points in the space-time. For two points $a = (x, y, t)$ and $a' = (x', y', t')$ in $Space \times Time$ we denote $\mathrm{dist}(a, b) = \max\{|x - x'|, |y - y'|, |t - t'|\}$. For $A, B \subset Space \times Time$ denote

$$\mathrm{dist}(A, B) = \min_{a \in A, b \in B} \mathrm{dist}(a, b).$$

The size (diameter) of a set $S \subset Space \times Time$ is defined as $\max_{a,b \in S} \mathrm{dist}(a, b)$

Let us fix now the constants q_n: we choose $C = 10000, q_0 = 3, q_n = Cn^2 q_{n-1}$. The exact values of this parameters are not very important. However, it is essential that $\sum(q_n/q_{n+1}) < \infty$.

Definition 1 (0-level islands and semi-islands).
A) We call a 0-level semi-island *every singleton containing a miracle.*
B) We call a 0-level island *any* 0*-level semi-island* S *such that* $\mathrm{dist}(S, M \setminus S) > q_1/5$.
C) The union of all 0*-level islands is denoted* M_0.

In general, M_0 is a proper subset of M.

Suppose that for all $k < n$ we have defined k-level semi-islands, k-level islands, and M_k. Then we define by induction semi-islands and islands of level n and the set M_n:

Definition 2 (n-level semi-islands and islands).
A) A nonempty set $S \subset M \setminus M_{n-1}$ *is called an* n-level semi-island *if*

(1) S *is of size at most* q_n*;*

(2) S *contains at least two disjoint* $(n-1)$*-level semi-islands.*

B) An n*-level semi-island* S *is an* n-level island *if*

(3) $\mathrm{dist}(S, M \setminus (S \cup M_{n-1})) > q_{n+1}/5$

C) We denote by M_n *the union of* M_{n-1} *and all* n*-level islands.*

Note that an n-level island is defined as an n-level semi-island that is isolated from other semi-island of level n and higher. Note also that any n-level semi-island contains at least 2^n miracles.

Definition 3 (birth time). *Let* S *be an* n*-level island. The* birth time *of* S *is* $\min\{t \,:\, \exists x, y\ (x, y, t) \in S\}$.

2.2 The structure of M

We say that an n-level semi-island $S \subset M$ is *minimal*, if any proper subset of S is *not* a n-level semi-island. Obviously, any n-level semi-island contains a subset that is a minimal n-level semi-island. Further we prove a statement concerning minimal semi-islands. In the following proposition we let $M_{-1} = \emptyset$ to simplify the notation.

Proposition 1. *For any m, for all $a \in M \setminus M_{m-1}$, the miracle a belongs to a* minimal *m-level semi-island S, which contains exactly 2^m miracles and is of size at most $q_m/3$.*

Proof: From Definition 2, condition (2), it follows that any n-level semi-island contains at least 2^n points. Thus, if some n-level semi-island contains exactly 2^n points, it is minimal.

We prove the proposition by induction. The inductive base $m = 0$ holds, as a 0-level semi-island is just a singleton, and $q_0 = 3$.

Assume that the statement is true for all $n < m$; let $a \in M \setminus M_{m-1}$. We must prove that a belongs to some minimal m-level semi-island. From the inductive hypothesis, there exists a minimal $(m-1)$-level semi-island S_1 such that $a \in S_1$; the set S_1 consists of 2^{m-1} points and is of size at most $q_{m-1}/3$.

Claim: *There exists a point $b \in M \setminus M_{m-2}$ such that $q_{m-1}/3 <$ $\operatorname{dist}(b, S_1) < q_m/4$.*

Assume the contrary is true, i.e., for any point $b \in M \setminus M_{m-2}$ if $\operatorname{dist}(b, S_1) \leq q_m/4$, then we have $\operatorname{dist}(b, S_1) \leq q_{m-1}/3$. In this case we can extend S_1 and get an $(m-1)$-level island, which implies a contradiction. More precisely, define

$$\tilde{S}_1 = \{b \in M \setminus M_{m-2} \ : \ \operatorname{dist}(b, S_1) \leq q_{m-1}/3\}.$$

The set $\tilde{S}_1$ is of size less than q_{m-1}. Hence, condition (1) from Definition 2 holds for $\tilde{S}_1$ with $n = m - 1$.

Further, $S_1 \subset \tilde{S}_1$, so (2) of Definition 2 is also true for $\tilde{S}_1$.

From our assumption we get that the distance between $\tilde{S}_1$ and $M \setminus (M_{m-2} \cup \tilde{S}_1)$ is at least $(q_m/4 - q_{m-1}/3)$, which is greater than $q_m/5$. Thus, condition (3) from Definition 2 is true for $\tilde{S}_1$. Hence, the set $\tilde{S}_1$ is a $(m-1)$-level island, $a \in M_{m-1}$, and we get a contradiction. The claim is proved.

From this Claim we get a cell $b \in M \setminus M_{m-2}$, which is at distance at least $q_{m-1}/3$ and at most $q_m/4$ from S_1. By inductive hypothesis, there exists a minimal $(m-1)$-level semi-island $S_2 \ni b$; the set S_2 consists of 2^{m-1} points and is of size at most $q_{m-1}/3$. As the distance between S_1 and b is at least $q_{m-1}/3$, the sets S_1 and S_2 are disjoint.

Set $S = S_1 \cup S_2$. By definition, S contains two disjoint $(m-1)$-level semi-islands and consists of $2^{m-1} + 2^{m-1} = 2^m$ miracles. Also, S is of size at most $(2/3 q_{m-1} + q_m/4) < q_m/3$. Thus, S is an m-level semi-island of size less than $q_m/3$. It is minimal, since S consists of 2^m points. $\square$

Definition 4. *Let u be a point in $Space \times Time$. Denote by $\epsilon_n(u)$ probability of the event that there exists at least one n-level semi-island S such that* $\mathrm{dist}(u, S) \leq q_{n+1}$.

Proposition 2. *There exists a $\gamma > 1$ such that for a small enough ϵ (from the definition of perturbation ξ) for any $u \in Space \times Time$ we have $\epsilon_n(u) \leq (\gamma\epsilon)^{2^n}$.*

Proof: Let us fix a point u in the space-time. We are interested in the probability of the event that there exists an n-level semi-island S such that $\mathrm{dist}(u, S) \leq q_{n+1}$. Note that such a semi-island S should be inside of the $2q_{n+1}$-neighborhood of u.

As any n-level semi-island contains a *minimal* n-level semi-island, it is enough to bound the probability of the following event: in the $2q_{n+1}$-neighborhood of u there exists a minimal n-level semi-islands.

We are going to count the total number of all minimal n-level semi-islands in the given neighborhood of u. Each n-level semi-island contains at least 2^n miracles. Hence, if the number of such semi-islands is L_n, then $\epsilon_n \leq L_n \cdot (\epsilon)^{2^n}$.

To make the arguments more clear, we shall count not the number of semi-islands but length of their descriptions. More precisely, we show that a minimal semi-island S in the given area can be uniquely identified (while the point u is known) by some description of length $\mathcal{O}(2^n)$. We suppose that any description is just a string of 0's and 1's. As there exist at most 2^l different descriptions of length l, there are at most $2^{\mathcal{O}(2^n)}$ different semi-island that have such a description. So, our bound for the length of descriptions implies

$$\epsilon_n \leq 2^{\mathcal{O}(2^n)} \cdot \epsilon^{2^n} < (\gamma\epsilon)^{2^n}$$

for some constant γ, which does not depend on ϵ and n.

We should define more formally what a complexity of a semi-island is. For the reader acquainted with the notion of Kolmogorov complexity [10] we could say that this is plain Kolmogorov complexity. But actually a more restrictive and simple definition is enough for our proof:

Definition 5. *Denote by $\mathcal{I}_n$ the set of all minimal n-level semi-islands in $(2q_{n+1})$-neighborhood of u. We shall say that there exist discriptions of length l_n for all n-level semi-islands in the given area, if there exists a surjective mapping*

$$\mathcal{F} : \{0,1\}^{l_n} \to \mathcal{I}_n.$$

Intuitively, $\mathcal{F}$ is a rule, which maps a description (a string of one's and zero's of length l_n) to the corresponding semi-island.

Lemma 1. *Let u be a point in the space-time. Then for all minimal semi-islands in $2q_{n+1}$-neighborhood of u there exist descriptions of length $l_n = D(\sum_{k \le n} \frac{\log k}{2^k})2^n$ for some constant D.*

In the proof we shall explain the description rule in intuitive terms. We believe it should be quite clear how to define the corresponding mapping $\mathcal{F}$ formally.

We prove this lemma by induction. The base is trivial: a 0-level semi-island is just a singleton. There are $\mathcal{O}(q_1^3)$ points in the $(2q_1)$-neighborhood of u, and we can provide descriptions of length $(3\log q_1 + \mathcal{O}(1))$.

Let us prove the inductive step. Let S be a minimal n-level semi-island in the q_{n+1}-neighborhood of u. From the definition of a semi-island it follows that S is a union of two disjoint semi-island of level $(n-1)$. Denote this islands S' and S''. It is not hard to see that we can choose two points u', u'' in the space-time such that

1. $\mathrm{dist}(u', S') \le q_n$,
2. $\mathrm{dist}(u'', S'') \le q_n$,
3. in each coordinate, the difference between u and u', and between u and u'' is a multiple of q_n.

Note that there are only $\mathcal{O}((q_{n+1}/q_n)^3) = \mathcal{O}((n+1)^6)$ possible positions for u' and u''.

To identify S given u, it is enough to identify u' and u'', and then identify S' given u' and S'' given u''. In a word, our description of S consists of 4 parts: description of u' given u, description of u'' given u, and descriptions of S' and S'' given u' and u'' respectively.

To describe each of two points u', u'', it is enough $\log((n+1)^6) + \mathcal{O}(1)$ bits of information. Further, by the inductive assumption, to identify S' given u' or S'' given u'' we need strings of length

$$(\sum_{k \le n-1} D\frac{\log k}{2^k})2^{n-1}.$$

In the whole, the description of S requires

$$\mathcal{O}(\log((n+1)^6)) + 2D(\sum_{k \le n-1} \frac{\log k}{2^k})2^{n-1} \le D(\sum_{k \le n} \frac{\log k}{2^k})2^n$$

digits (if D is taken large enough). The lemma is proven.

To prove the proposition it remains to note that the series $\sum \frac{\log k}{2^k}$ converges. Thus, any minimal n-level semi-island S in q_{n+1}-neighborhood of u has a description of length $O(2^n)$, and we are done.

$\square$

Remark 1. The fact that $Space \times Time$ has dimension 3 is not important for the proof above. The same argument would work for any finite dimension.

Remark 2. Proposition 2 implies that $\sum_{i<\infty} \epsilon_i = \mathcal{O}(\epsilon)$ if ϵ is small enough.

Corollary 1. *If ϵ is small enough, with probability 1 every miracle belongs to some n-level island.*

Proof: Let $u \in M$. By Proposition 1 there are two possibilities: either (i) $u \in M_n$ for some $n \geq 0$, i.e., u belongs to some n-level island, or (ii) u belongs to an n-level semi-island for every $n \geq 0$. From Proposition 2 it follows that (ii) holds with probability 0, since $\epsilon_n \to 0$ as $n \to \infty$. $\square$

We shall say that the set of miracles M is *standard*, if each miracle belongs to some n-level island.

2.3 Evolution of an n-level island

Every alive cell is alive because it is a consequence of some miracles. Let us define this more formally.

Definition 6. *An explanation path from an alive point $a \in Space \times Time$ to an n-level island S is a sequence $a_0, a_1, \ldots, a_m$ of alive points in $Space \times Time$ such that*

1. $a_0 = a,\ a_m \in S$
2. *for any $k < m$ if a_k has coordinates (x, y, t) then a_{k+1} is one of three points $(x, y, t-1)$, $(x+1, y, t-1)$, $(x, y+1, t-1)$.*

Note that some a_i in the sequence above might belong to other islands but S.

If there exists an explanation path from an alive point a to an n-level island S, we say that a is a consequence of S.

Obviously, for a *standard* set of miracles M any alive point is a consequence of some island of miracles (or, maybe, of many islands).

Remark 3. Let a sequence of points $a_0, a_1, \ldots, a_m$ be an explanation path, and $a_i = (x_i, y_i, t_i)$ for $i \leq m$. Then $t_i - t_j \geq \max\{x_j - x_i, y_j - y_i\}$ for any $i < j$.

There may be more than one explanation path from a point a to an island S. Further we select explanation paths with some special properties.

Definition 7 (Space-greedy explanation path). *Let $a_0, \ldots, a_m$ be an explanation path from an alive point $a = a_0$ to some island S. This path is called* space-greedy *if the following conditions hold for any $i < m$:*

– *$a_i \notin S$.*
– *Let $a_i = (x, y, t)$. This point is a consequence of S but not its member. Hence, at least two of points $(x, y, t-1)$, $(x+1, y, t-1)$, $(x, y, t+1)$ are alive, and, moreover, at least one of them is a consequence of S. For a space-greedy path, if $(x+1, y, t-1)$ or $(x, y+1, t-1)$ is a consequence of S, then a_{i+1} must be one of these two points. Only if none these points is a consequence of S, $a_{i+1} = (x, y, t-1)$.*

And two similar definitions:

Definition 8 (South-most explanation path). *Let a sequence of points $a_0, \ldots, a_m$ be an explanation path from an alive point $a = a_0$ to an island S ($a_m \in S$). This path is called* south-most *if for any $i < m$ the following conditions hold:*

– *$a_i \notin S$.*
– *If one of points $(x, y, t-1)$, $(x+1, y, t-1)$ is alive, then a_{i+1} must be one of these two points. Otherwise $a_{i+1} = (x, y+1, t-1)$,*

Definition 9 (West-most explanation path). *Let a sequence $a_0, \ldots, a_m$ be an explanation path from an alive point $a = a_0$ to some island S ($a_m \in S$). This path is called* west-most *if for any $i < m$ the following conditions hold:*

– *$a_i \notin S$,*
– *If one of points $(x, y, t-1)$, $(x, y+1, t)$ is alive, then a_{i+1} is one of these two points. Only otherwise $a_{i+1} = (x+1, y, t-1)$.*

Note that if u is a consequence of some n-level island S, then there exist a space-greedy, a west-most and a south-most explanation paths from u to S.

Definition 10. *A point $a \in Space \times Time$ is called a* proper consequence *of an n-level island S, if the following conditions hold*

1. *a is not a consequence of any island of level higher than n.*
2. *if a is a consequence of another n-level island S', then the birth time of S is not greater than the birth time of S'.*

Lemma 2. *Assume the set of miracles M is standard, $a \in M$, and a is a proper consequence of an n-level island S. Assume a is also a consequence of another n-level island S'. Then for any consequence b of S' we have $\mathrm{dist}(b, S) > 10q_n$.*

Proof of lemma: Let $b = (x, y, t)$ be a consequence of S', and assume $\mathrm{dist}(b, S) \leq 10q_n$. Denote by t_0 the birth time of S. Then

$$t - t_0 \leq 11q_n. \tag{1}$$

The point b is a consequence of S', so there exists an explanation path $b_0, b_1, b_2, \dots, b_m$ from b to S' ($b = b_0$ and b_m is point in S'). Let $b_m = (x_m, y_m, t_m)$. As $\mathrm{dist}(b, S) \leq 10q_n$ and $\mathrm{dist}(S, S') > q_{n+1}/5$ (the distance between any two n-level islands must be large), we get $\mathrm{dist}(b, b_m) > q_{n+1}/5 - 10q_n$. Hence, by Remark 3,

$$t - t_m > q_{n+1}/5 - 10q_n. \tag{2}$$

From inequalities (1) and (2) we have $t_m < t_0$, so the birth time of S' is less then the birth time of S. Hence, by Definition 10, item (2), a is not a proper consequence of S. Thus, we get a contradiction. $\square$

Theorem 1. *Suppose that the set of miracles M is standard. Let a be a proper consequence of an n-level island S. Then $\mathrm{dist}(a, S) < 8q_n$. Consequently, for any point $b \in S$ the distance between a and b is less than $9q_n$.*

Proof of theorem: We prove the theorem by induction. Inductive basis $n = 0$ is trivial. Let us deal with inductive step. We should prove that an explanation path from S to its proper consequence a cannot be too long. It is enough to prove that if a is a proper consequence of S and $\mathrm{dist}(a, S) < 10q_n$, then we have $\mathrm{dist}(a, S) < 8q_n$.

Lemma 3. *Let a be a consequence of S. Let $P = (a_0, a_1, \dots, a_k)$ be a space-greedy explanation path from $a = a_0$ to some $a_k = a' \in S$. Let $a = (x, y, t)$ and $a_k = (x', y', t')$. Denote $T = t - t'$ and $L = (x' - x) + (y' - y)$. Suppose $T > 5q_{n-1}$. Then $L \geq 9T/10$.*

Informally, this lemma says that the hypotenuse of Toom's triangle around S moves to the south-west with average speed greater than $9/10$.

Proof of lemma: Obviously, every next point in the path P has the time coordinate less by 1, so $T = k$. Further, any a_i either has the same space coordinates as a_{i+1}, or one of its space coordinates is shifted by 1. To get the Lemma we should prove that there are at most 10% of such points a_i in the path P, that a_i and a_{i+1} have the same space coordinates.

Let $a_i = (x_i, y_i, t_i)$ and $a_{i+1} = (x_i, y_i, t_i - 1)$ have the same space coordinates. The points $(x_i + 1, y_i, t_i - 1)$ and $(x_i, y_i + 1, t_i - 1)$ are not consequences of S, because P is space-greedy. As a_i is alive, two variants are possible:

1. a_i is a miracle and, hence, it belongs to some island S';
2. at least one of points $(x_i + 1, y_i, t_i - 1)$, $(x_i, y_i + 1, t_i - 1)$ is alive and, hence, is a proper consequence of some island $S' \neq S$.

In both cases we say that S' *supports* the path P at the point a_i.

As a is a proper consequence of S, from Lemma 2 it follows that the level of S' is less than n.

For each number $i < k$ such that the point a_i has the same space coordinate as a_{i+1}, we fix an island $S' = S'(a_i)$ as above. We should answer two questions:

– How many points a_i of the path P can support one k-level island S'?
– How many k-level islands can support the path P at different points?

Answer to the first question is simple: by the inductive hypothesis of Theorem 1, it is less than $9q_{k-1}$ (all proper consequences of a k-level islands S' are at the distance at most $9q_k$ from any point in S').

To answer the second question some calculations are required. Assume that S' and S'' are two k-level islands that support the path P at points a_i and a_j respectively. By the inductive hypothesis, $\mathrm{dist}(a_i, S') < 8q_k$ and $\mathrm{dist}(a_j, S'') < 8q_k$. The same time, the distance between any two k-level islands is lower-bounded: $\mathrm{dist}(S', S'') > q_{k+1}/5$. Hence, the distance between a_i and a_j is quite large. At least, $\mathrm{dist}(a_i, a_j) > q_{k+1}/10$ (a very rough bound). From Remark 3 we get $|t_i - t_j| = |i - j| > q_{k+1}/10$. Upperbound the number of all points a_i on the path that are supported by k-level islands: it is less than $\lceil T/(q_{k+1}/10)\rceil \cdot 9q_k$. Sum up this value for all $k < n$:

$$\sum_{k<n} \lceil 10T/q_{k+1}\rceil \cdot 9q_k \le (1 + 1/5) \cdot 90 \sum_{k<n} Tq_k/q_k < 120T \sum_{k<\infty} (q_k/q_{k+1})$$

(the first inequality follows from the condition $T > 5q_{k+1}$ for all $k < n$). Further, in the definition of q_n the constant C was chosen so that

$$\sum_{k=0}^{\infty} (q_k/q_{k+1}) \le 1/2000.$$

Lemma follows from this bound immediately.

Lemma 4. *Let a be a consequence of S, and $P = (a_0, a_1, \ldots, a_k)$ be a southmost explanation path from $a = a_0$ to some $a' = a_k \in S$. Let $a = (x, y, t)$ and $a_k = (x', y', t')$, and denote $T = t - t'$, $L' = (y' - y)$. Suppose that $T > 5q_n$. Then $L' \le T/10$.*

Informally, this lemma says that the horizontal leg of Toom's triangle around S moves to the south with average speed less than $1/10$.

Proof of lemma: very similar to the proof of Lemma 3. We are interested in points a_i on the path P such that a_i and a_{i+1} have different Y-coordinates. We should prove that there are at most 10% of such points in P.

Let $a_i = (x_i, y_i, t_i)$ and $a_{i+1} = (x_i, y_i + 1, t_i - 1)$. Then the points $(x_i + 1, y_i, t_i - 1)$ and $(x_i, y_i, t_i - 1)$ are not consequences of S, because P is south-most. Hence, either a_i is a miracle (and belongs to some island S'), or at least one of point $(x_i + 1, y_i, t_i - 1)$, $(x_i, y_i, t_i - 1)$ must be alive and be a proper consequence of some islands $S' \neq S$. In these cases we say that S' *supports* the path P at the point a_i.

How many points a_i can support one k-level island S'? By the inductive hypothesis of Theorem 1, the number of such points is less than $9q_k$.

How many k-level islands can support the path P at different points? Assume that S' and S'' are two k-level islands that support P at points a_i and a_j respectively. By inductive hypothesis, the distances $\mathrm{dist}(a_i, S')$ and $\mathrm{dist}(a_j, S'')$ are both less than $8q_k$. The same time $\mathrm{dist}(S', S'') > q_{k+1}/5$. Hence, $|i - j| > q_{k+1}/10$.

The number of all points a_i supported by k-level islands is at most $\lceil T/(q_{k+1}/10)\rceil \cdot 9q_k$. Sum up this value for all $k < n$:

$$\sum_{k<n} \lceil 10T/q_{k+1}\rceil \cdot 9q_k \leq (1 + 1/5) \cdot 90T \sum_{k<n} q_k/q_{k+1} \leq T/10.$$

Lemma is proved.

Also a statement symmetrical to Lemma 4 holds:

Lemma 5. *Let a be a consequence of S, and $P = (a_0 = a, a_1, \ldots, a_k)$ be a west-most explanation path from a to S, where $a = (x, y, t)$ and $a_k = (x', y', t')$. Let $T = t' - t$ and $L'' = (x' - x)$. If $T > 5q_n$ then $L'' \leq T/10$.*

Informally, this lemma says that the vertical leg of Toom's triangle around S moves to the west with average speed less than $1/10$.

Now we can prove the theorem. The idea is simple: for each time t draw Toom's triangle around the set of all consequences of the island S. The lemmas above imply that as the time coordinate increase, the hypotenuse of this triangle goes to the south-west with average speed at least $9/10$, and the vertical and horizontal legs go to the west and south respectively with small average speed (at most $1/10$). Hence, the triangle must shrink in time $\mathcal{O}(q_n)$.

The same idea can be expressed in other terms. Let some point a is a consequence of an error island S. Then there are south-most, west-most and space-greedy explanation paths from a to S. On one hand, these three paths diverge when we go to the past; on the other hand, the ends of all three paths must be inside S, see Fig. 3. Hence, the distance in time and space between a and S

cannot be to large, so these explanation paths have no possibility to diverge too far.

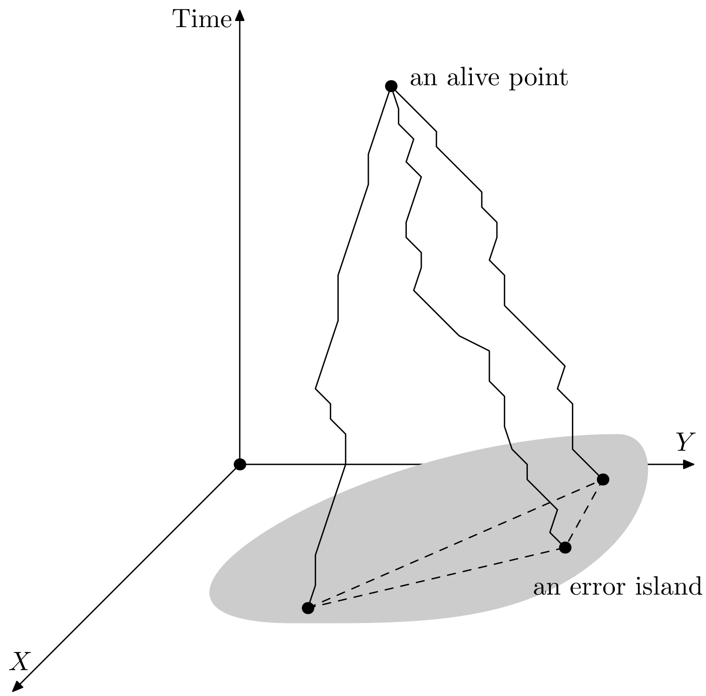

Fig. 3. Three explanation paths from an alive point to an error island

Let us do the precise computations. Denote by t_{first} and t_{last} the minimal and the maximal times of miracles in S; denote by $x_{\min}$, $x_{\max}$, $y_{\min}$, $y_{\max}$ the minimal and the maximal space coordinates of miracles in S respectively.

Let $a = (x, y, t)$ be a consequence of S. First, we prove that $t - t_{last} < 8q_n$. If $t - t_{last} \leq 5q_n$, there is nothing to prove. Assume that contrary and employ the lemmas above. From Lemma 3 it follows that for a space-greedy explanation path from a to $a' = (x', y', t') \in S$ the difference $(x' + y') - (x + y)$ is quite large comparative to $(t - t')$. More exactly, we have $(x_{\max} - x) + (y_{max} - y) > 9/10(t - t_{last})$. Further, from Lemma 4 the south-most explanation path goes from a to $a'' = (x'', y'', t'') \in S$ such that $(y'' - y)$ is quite small comparative to $(t - t'')$. More precisely, $(y_{min} - y) < 1/10(t - t_{first})$. Similarly, from Lemma 5 we get $(x_{min} - x) < 1/10(t - t_{first})$. It is not hard to check that these three inequality above imply $t - t_{last} < 5q_n$ (a very rough bound).

Thus, we have proved that for any proper consequence $a = (x, y, t)$ of S the bound $t - t_{last} < 5q_n$ holds. Let us fix any explanation path from a to S. Denote by $a' = (x', y,', t') \in S$ the last point in this explanation path. Then

$$t - t' \leq (t - t_{last}) + (t_{last} - t_{first}) \ll 8q_n.$$

From Remark 3, $x' - x \leq t - t'$ and $y' - y \leq t - t'$. Hence, $\mathrm{dist}(a, S) < 8q_n$. $\square$

Corollary 2. *Any point $a \in Space \times Time$ is dead with large probability $\hat{\epsilon}$ ($\hat{\epsilon} \to 1$ as $\epsilon \to 0$).*

Proof of corollary: First of all, we may assume that the set of all miracles M is standard (Corollary 1).

Let a be an alive point. As M is standard, from Lemma 2 it follows that a is a proper consequence of some island S. From Theorem 1 it follows that the point a is in the $8q_n$-neighborhood of S.

From Proposition 2 we get that probability of the event 'a is on the distance less then $8q_n$ from some n-level island' is less than $\epsilon_n \leq (\gamma\epsilon)^{2^n}$. Hence, a is alive with probability less than the sum $\sum\limits_{i<\infty} \epsilon_i = \mathcal{O}(\epsilon)$, see Remark 2.

The Corollary above implies a more general statement. Until now, we allowed only 'one-way' noise: a dead cell could become spontaneously alive, but not visa-versa. Let us consider a more traditional model. Let we have a 2-D cellular automaton, each cell has two states. At each cell normally Toom's rule is applied, but with small probabilities a cell can randomly change its state (from 'alive' to 'dead' or from 'dead' to 'alive'). Such a probabilistic automaton is usually called *a small perturbation of Toom's rule*. Again, we suppose that random transitions at different cells and at different moments of time are independent. Combining the result above with the standard arguments [11], we get that any such automaton has at least two different invariant measures: 'most cells are alive' and 'most cells are dead'.

3 Toom's rule on torus.

In this section we discuss the behavior of Toom's automaton on a finite space-time. Let p be a positive integer, and $Space$ be the torus $\mathbb{Z}_p \times \mathbb{Z}_p$, where $\mathbb{Z}_p$ is the set of integers modulo p. The time scale will be also finite: $Time = \{0, \ldots, T\}$. The definition of Toom's faulty automaton from Section 2 can be obviously used for this finite variant of $Space \times Time$. Moreover, as any small area on the torus is equivalent to an area on the plane, the behavior of Toom's automaton on a torus and on the plane is quite similar. Further we explain how the arguments from Section 2 can be applied in the new situation.

3.1 Notation and definitions

Define random perturbation $\xi(a)$ and the evolution of the automaton $s(x, y, t)$ word for word as in Section 2. The only difference is that now x, y run over $\mathbb{Z}_p$ and $t \in \{0, \ldots, T\}$.

For $u, u' \in \mathbb{Z}_p$ denote

$$|u - u'|_p = \min_{k \in \mathbb{Z}} |u - u' + kp|.$$

Define the distance between points in the space-time as follows: for $a = (x, y, t)$ and $a' = (x', y', t')$ in $Space \times Time$

$$\mathrm{dist}(a, b) = \max\{|x - x'|_p, |y - y'|_p, |t - t'|\}.$$

As usually, for $A, B \subset Space \times Time$ denote

$$\mathrm{dist}(A, B) = \min_{a \in A, b \in B} \mathrm{dist}(a, b),$$

and call *size of a set* $S \subset Space \times Time$ its diameter defined as $\max\limits_{a,b \in S} \mathrm{dist}(a, b)$.

The old definition of the n-level semi-islands and the n-level islands can be used now for the finite variant of space-time (employing the defined above notion of distance on the finite $Space \times Time$).

3.2 Adopting the proof for a torus.

All arguments of Section 2 work for n-level islands in the finite space-time if only their size is small comparative to the size of torus. We shall suppose that all islands of level lower than n_0 are small enough:

Definition 11. *Let us fix the size p of the space-time. Denote by n_0 the maximal integer such that $q_{n_0} \le p$.*

The following analogs of Proposition 1 and Proposition 2 hold for the finite case:

Proposition 3. *For any $m < n_0$, for all $a \in M \setminus M_{m-1}$, the miracle a belongs to a* minimal *(by inclusion) m-level semi-island S. Such a set S contains exactly 2^m miracles and is of size at most $q_m/3$.*

Proposition 4. *There exists $\gamma > 1$ such that for small enough ϵ (from the definition of perturbation ξ) for any $u \in Space \times Time$ and any $n < n_0$*

$$\epsilon_n(u) \le (\gamma\epsilon)^{2^n},$$

where ϵ_n is defined as in Definition 4.

The proofs are exactly the same as in the infinite case.

Corollary 3. *Let*

$$Space \times Time = \mathbb{Z}_p \times \mathbb{Z}_p \times \{0, \dots, T\}.$$

Then every miracle belongs to an n-level island for some $n < n_0$ with probability $(1 - \epsilon_{n_0}) \geq 1 - (\gamma\epsilon)^{2^{n_0}}$ (for some $\gamma > 1$, and ϵ from the definition of perturbation ξ).

Proof: Let $a \in M$. By Proposition 3, if $a \notin M_n$ for all $n < n_0$, then a belongs to some n_0-level semi-island. But probability of this event is at most ϵ_{n_0}. $\square$

We say that the set of miracles on the finite space-time is *standard* if each miracle belongs to an n-level island for some $n < n_0$.

Theorem 2. *Let*

$$Space \times Time = \mathbb{Z}_p \times \mathbb{Z}_p \times \{0, \dots, T\}.$$

Suppose that the set of miracles M is standard. Let a be a proper consequence of an n-level island S. Then $\mathrm{dist}(a, S) < 8q_n$.

This theorem can be proved by the same arguments as Theorem 1 in Section 2. From Theorem 2 we get a corollary:

Corollary 4. *Let*

$$Space \times Time = \mathbb{Z}_p \times \mathbb{Z}_p \times \{0, \dots, T\}.$$

Then each point $a \in Space \times Time$ is alive with probability $p > 1 - p^2 T \epsilon_{n_0} - \mathcal{O}(\epsilon)$, if ϵ is small enough.

Proof: First of all note that the total number of points in $Space \times Time$ with a *positive* time coordinate is $p^2 T$. Hence, with probability at least

$$(1 - p^2 T \epsilon_{n_0})$$

there is no n_0-level semi-islands in $Space \times Time$. Further, if there is no n_0-level semi-islands, then each miracle belongs to some n-level island for $n < n_0$, i.e., the set of miracles is standard. Then every alive point is a proper consequence of some n-level island. From Theorem 2 it follows that a proper consequence of an n-level island S must be in $8q_n$-neighborhood of S. Hence, each point is alive with probability at most

$$\sum_{i<n_0} \epsilon_i = O(\epsilon).$$

$\square$

Remark 4. We defined $n_0 = n_0(p)$ as the maximal integer such that $q_{n_0} < p$. It is not hard to see that for a fixed ϵ the value $\epsilon_{n_0} = \epsilon_{n_0}(p)$ tends to zero (as $p \to \infty$) faster than any polynomial in p. Thus, if we want to guarantee that each point in space-time is alive with probability $\mathcal{O}(\epsilon)$, we can let T grow faster than any polynomial in p.

4 Implementing fault-tolerant computations on a 3D cellular automaton

In this section we use the technique developed above, to prove the result from [4]: we explain how to construct a 3-D cellular automaton, which simulates a given 1-D cellular automaton in spite of perturbation (i.e., assuming that any cell at any moment with a small probability can randomly change its state). We assume that in the simulated automaton the state of a cell depends on its own state and the states if its closest neighbors on the previous step.

In our model, we have an infinite space $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ (if we want to simulate an infinite 1-D array of cellular automata) or $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ (if we simulate a finite 1-D array). There is a fixed finite automaton in each cell of the space. Each vertical column in the space, i.e., each family of cells

$$C_{u,v} = \{(u, v, z) \in Space\}$$

is supposed to simulate a 1-D array of cellular automata. One cell of our automaton corresponds to one cell of the simulated automaton. Denote by $s(x, y, z, t)$ the state of the cell with coordinates (x, y, z) at time t.

In the beginning, all columns should be synchronized and represent the initial configuration of the simulated array. If there is no random mistakes, all columns should synchronously simulate the computation of the simulated array. In this case for each $t > 0$ the state $s(x, y, z, t)$ is a function of its closest neighbors in the column:

$$s(x, y, z, t + 1) = Trans(s_{-1}, s_0, s_{+1}),$$

where

- $s_{-1} = s(x, y, z - 1, t)$,
- $s_0 = s(x, y, z, t)$,
- $s_{+1} = s(x, y, z + 1, t)$,

and $Trans$ is the transitions rules of the simulated automaton. Thus, if there is no faults, we just simulate a bunch of 1-D arrays, and they all are working synchronously.

To make the simulation working in presence of faults, each cell should observe also its neighbors in nearby columns. We compose the transition mapping F with Toom's rule. More exactly, the following rule is used:

$$s(x, y, z, t+1) = Trans(s_{-1}, s_0, s_{+1}),$$

where

- s_{-1} is majority of the triple $s(x, y, z-1, t)$, $s(x+1, y, z-1, t)$, $s(x, y+1, z-1, t))$,
- s_0 is majority of the triple $s(x, y, z, t)$, $s(x+1, y, z, t)$, $s(x, y+1, z, t))$,
- s_{+1} is majority of the triple $s(x, y, z+1, t)$, $s(x+1, y, z+1, t)$, $s(x, y+1, z+1, t))$.

We assume that random faults in all cells and any moments of time are independent. Our aim is to prove that if probability of a random mistake at any given point and any time is small enough, then for all x, y, z, t the state $s(x, y, z, t)$ with high probability presents the correct value of the cell z of the simulated array at moment t.

4.1 Dead and alive cells again

To prove correctness of the automaton above we should investigate the behavior of the 'spoiled' cells, i.e., those cells whose state is not correct due to random faults. We define a very simple automaton with only two states: 'alive' and 'dead'. We define it so that at any moment the set of all alive points of the new automaton covers the set of all 'spoiled' cells of the original automaton. In other words, if the original automaton has a spoiled cell at (x, y, z) at moment t, then the new automaton (being disturbed with the same faults) must have an alive point at (x, y, z) at the same time t. Note that converse is not true, i.e, the new automaton can have an alive cell (x, y, z) at moment t even though the corresponding cell of the original automaton is not spoiled.

Let us describe the new automaton. Initially, all cells of the new automaton are dead. The cells are updated by the following rule: *The cell* (x, y, z) *is alive at moment* $(t+1)$ *if at least one of the majorities* s_{-1}, s_0, s_{+1}, *where again*

- s_{-1} is majority of the triple $s(x, y, z-1, t)$, $s(x+1, y, z-1, t)$, $s(x, y+1, z-1, t))$,
- s_0 is majority of the triple $s(x, y, z, t)$, $s(x+1, y, z, t)$, $s(x, y+1, z, t))$,
- s_{+1} is majority of the triple $s(x, y, z+1, t)$, $s(x+1, y, z+1, t)$, $s(x, y+1, z+1, t))$.

Besides the deterministic rule above, at any moment any dead cell can be randomly made alive. As in the previous sections, we call these events 'miracles'. Miracles at different cells and moment of time are independent.

More precisely, let us define the perturbation as a random function $\xi(x, y, z, t)$, which has two values (0 and 1). We suppose $\xi(x, y, z, t) = 1$ with a small probability $\epsilon > 0$, and values of this function at different points of space-time are independent. If $\xi(x, y, z, t) = 1$, we make the cell (x, y, z) of the automaton alive at time t; otherwise (if $\xi(x, y, z, t) = 0$) we apply the deterministic rule above.

We will prove that if ϵ is small enough, then any cell at any time is dead with high probability. Clearly, this result implies that the fault tolerant simulation of a 1D array of cellular automata (defined above) is adequate (i.e., each cell at any moment with high probability has a correct value).

The proof of the result above follows the same plan as our proof of Theorem 1 and Theorem 2. We should just update the definitions to deal with 4-D space-time. Further we explain how the required modifications can be done.

4.2 Adopting the arguments for 3-D space

Now $Space$ is the set $\mathbb{Z}^3$ or $\mathbb{Z}_p^3$. Proposition 1 holds for the new definitions (the old proof is valid). The same is true for Proposition 2 (see Remark 1). The only non-trivial modifications are required in the proof of Theorem 1. We should adopt the definition of explanation paths to 3-D space. This can be done as follows.

Definition 12. *An explanation path from an alive point $a \in Space \times Time$ to an n-level island S is a sequence $a_0, a_1, \ldots, a_m$ of points in $Space \times Time$ such that*

1. $a_0 = a,\ a_m \in S$
2. *for any $k < m$ if a_k has coordinates (x, y, z, t) then a_{k+1} is one of the points $(x, y, z', t-1)$, $(x+1, y, z', t-1)$, $(x, y+1, z', t-1)$, where $z' \in \{z-1, z, z+1\}$ (nine variants in total).*

If there exists an explanation path from an alive point a to an n-level island S, we say that a is a consequence of S.

Definition 13. Space-greedy explanation path: *Let $a_0, \ldots, a_m$ be an explanation path from an alive point $a = a_0$ to some island S. This path is called* space-greedy *if the following conditions hold for any $i < m$:*

- $a_i \notin S$.

– *Let* $a_i = (x, y, z, t)$. *This point is a consequence of* S *but not its member. Hence, at least one of points* $(x, y, z', t-1)$, $(x+1, y, z', t-1)$, $(x, y, z', t+1)$ $(z' \in \{z-1, z, z+1\})$ *is alive and at least one of them is a consequence of* S. *For a space-greedy path, if one of points* $(x+1, y, z', t-1)$ *or* $(x, y+1, z', t-1)$ *is a consequence of* S*, then* a_{i+1} *must be one of these two points. Only if none these points is a consequence of* S, $a_{i+1} = (x, y, z', t-1)$.

Definition 14. South-most explanation path*: Let* $a_0, \ldots, a_m$ *be an explanation path from an alive point* $a = a_0$ *to an island* S *(*$a_m \in S$*). This path is called* south-most *if for any* $i < m$ *the following conditions hold:*

– $a_i = (x, y, z, t) \notin S$.
– *If one of points* $(x, y, z', t-1)$, $(x+1, y, z', t-1)$ $(z' \in \{z-1, z, z+1\})$ *is alive, then* a_{i+1} *must be one of these two points. Otherwise* $a_{i+1} = (x, y+1, z', t-1)$,

Definition 15. West-most explanation path*: Let* $a_0, \ldots, a_m$ *be an explanation path from an alive point* $a = a_0$ *to some island* S *(*$a_m \in S$*). This path is called* west-most *if for any* $i < m$ *the following conditions hold:*

– $a_i = (x, y, z, t) \notin S$,
– *If one of points* $(x, y, z', t-1)$, $(x, y+1, z', t)$ $(z' \in \{z-1, z, z+1\})$*is alive, then* a_{i+1} *is one of these two points. Only otherwise* $a_{i+1} = (x+1, y, z', t-1)$.

Based on this definitions, we can apply the arguments from the proof of Theorem 1 given in Section 2. We omit the details.

Note that in the finite case the computation on a zone p can be simulated during super-polynomial time, see Remark 4.

5 Conclusion

In this paper we presented a detailed proof of Toom's theorem, which says that Toom's 2-dimensional cellular automaton has stable nontrivial global state. We showed how the same ideas help to implement reliable computations based on a 3-dimensional array of faulty cellular automata. We proved that our construction allows to simulate any polynomial time algorithm on faulty cell automata, and the result is correct with probability $1 - O(\epsilon)$ if any cell at any step of computation is corrupted with probability ϵ. We stress that the construction works for any small enough ϵ, for any size of the array (and polynomial number of steps of the computation).

In our opinion (which is, of course, quite subjective), the new proof is more easy to understand than the previous ones. More specifically, we understand better the consequences of faults on computations.

We hope that our work will help to use these methods in other problems concerning cellular automata and related areas.

References

1. Gács P: Lectures on Reliable Cellular Automata. http://www.cd.bu.edu/faculty/gacs/papers/ca-lectures.pdf
2. Toom A: Stable and Attractive Trajectories in Multicomponent Systems, *Multicomponent systems, Advances in Probability*. 1980; 6: 549–576.
3. Berman P, Simon J: Investigations of fault-tolerant networks of computers. 20th STOC. 1988: 66–77.
4. Gács P, Reif J: A simple three-dimensional real-time reliable cellular array, *Journal of Computer and System Sciences*. 1988; 36,2: 125–147.
5. Gács P: A New Version of Toom's Proof. *Tech. report*, Department of Computer Science, Boston University, TR-95-009, Boston. MA 02215, 1995.
6. Gács P: A Toom rule that increases the thickness of sets. *Journal of Statistical Physics*. 1990; 59, 1/2: 171–193.
7. Gács P: Self-Correcting Two-dimensional Arrays, *Randomness in Computation*, Advances in Computing Research (a scientifi c annual). JAI Press, Greenwich, Comm. 1989; 5: 223–326.
8. Gács P: Reliable Cellular Automata with Self-Organization. *Journal of Statistical Physics*. 2001; 103, 1/2: 45–267.
9. Grey L-F: A Reader's Guide to Gács' "Positive rates" Paper. *Journal of Statistical Physics*. 2001; 103,1/2: 1–44.
10. Li M, Vitanyi P: Kolmogorov Complexity and its Applications, 2nd Ed., Springer-Verlag, New York, 1997.
11. Toom A: Contours, convex sets, and cellular automata. Text of a course at the 23-rd Colloquium of Brazilian mathematicians. IMPA, 2001.