Clustering and Robustness in Networks

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Extended Abstract.
The robustness of a network is defined as the integrity of the network to attack through the deletion of network nodes. Such attacks can be random or targeted: in the former case nodes are selected for removal at random; in the latter one usually chooses to remove the most highly connected nodes at each stage, either systematically or with some enhanced probability. As nodes are removed the size of the largest connected component is tracked. In a robust network the size of this component decreases slowly as nodes are removed; a fragile network rapidly breaks into sub-networks.

The connectivity distribution of a network p_k is the probability that a randomly chosen node of the network has degree k, that is has k links to or from the node. Ever since Albert *et al* [1] showed that the robustness of networks depends on the distribution of connectivity and the mode of attack, random or targeted, there have been numerous attempts [2, 3, 4] to understand the relationship between the architecture of a network and its robustness.

The impression from the seminal paper of Albert *et al* [1], which compares Erdös-Rényi random networks [5], with a Poisson degree distribution, with a class of scale-free networks [6], with their power law distribution of connectivity, is that robustness is associated with the connectivity distribution.

Newman [7] argues that the robustness to node removal is related to what is referred to as the assortative behaviour, or assortativity, of the network. This is defined as the preference for nodes of large degrees of connectivity to be attached to each other. It is measured by the assortative coefficient, r. To define r, let e_{ij} be the joint probability distribution of the degrees of the nodes at the ends of a randomly chosen link, not counting this link itself in the nodal degrees [8]. Then r ($-1 \le r \le 1$) is given by

$$r \propto \frac{\sum_{ij} ij(e_{ij} - q_i q_j)}{\left(\sum_k k^2 q_k - \sum_k k q_k\right)},$$

where the normalised 'remaining degree' distribution [4, 6] q_k is

$$q_k = \frac{(k+1)\, p_{k+1}}{\sum_j j p_j}.$$

The coefficient r is positive for assortative networks and negative otherwise. Thus sociological networks ($r > 0$) appear to be more robust than biological or physical networks ($r < 0$). However, since both random and scale-free networks have the same assortativity but react differently to node removal, the assortative behaviour cannot be the only criterion for the robustness of a network.

By investigating networks having the same assortativity but different degrees of clustering, we show here that robustness depends also on the cluster coefficient of a network.

Clustered networks
The cluster coefficient is defined as the degree of clustering of a node averaged over all the nodes of the network. It is given by

$$\langle C \rangle = N^{-1} \sum_i \frac{T_i}{2k_i(k_i - 1)},$$

with T_i the number of triangle from node i, k_i the degree of connectivity of node i, and N the number of nodes in the network.

We begin by constructing networks with a fixed connectivity, k, and assortativity, of an ordered network of nodes arranged on a circle. We vary the cluster coefficient via a 'cross-rewiring' operation. It leads to a sequence of networks with decreasing degrees of clustering. We call these networks clustered k-regular networks, or kC-networks for short.

Robustness
We extract networks along the rewiring process and test their robustness to the removal of nodes at random or to the continued targeted (systematic) removal of the most highly connected nodes. In the latter case, if several nodes have the same degree of connectivity, one is targeted at random.

In the case of an attack on the most connected nodes of a kC-network, the critical value at which the network breaks down increases as the clustering coefficient diminishes. This is somewhat counter-intuitive as one might expect that a network with larger cluster coefficient provides more redundancies of pathways between any pair of nodes. The results show the opposite, because rewiring links to form clusters reduces the number of links associated with the large scale resilience of the network.

This behaviour is confirmed when the nodes are removed at random: kC-networks with a very low cluster coefficient are more resilient to node failure than their highly clustered counterparts.

Aside from the fact that these networks are more or less robust according to the clustering coefficient, they always break down when the most connected nodes are removed. On the contrary, the random removal of nodes from a network with a small value of C (<0.275) does not break it down but leads to a steady erosion of the largest component.

Note there exists a value of C ($0.370 < C < 0.470$) for which the network behaves in the same way to random or targeted removal.

Random and scale-free networks
Random and scale-free network present a range of connectivity that requires us to set a constraint on the cross-rewiring operation in order to maintain the assortativity constant.

We study the behaviour of random and scale-free networks for various clustering coefficients as above. The random network is an Erdös and Rényi network, where pairs of nodes are connected with probability, p. The scale-free network is generated using preferential attachment in the context of growing networks. By construction, both networks have an assortative coefficient r = 0.

For both networks, there is still an effect of the clustering coefficient on their resilience. Networks with smaller clustering coefficients are more robust than their counterparts, in either random or targeted removal of the nodes. The main difference from our previous case is that the random removal of nodes no longer has a critical effect: the main connected component loses only few nodes at a time and does not fall apart.

As observed previously [1], the critical values at which the network breaks down is lower for the scale-free network than for the random one.

Along with assortativity, the clustering coefficient is an important parameter in determining the architectural robustness of networks. Networks with smaller clustering coefficient are more robust than networks with larger one. As a consequence, the small world property, which is associated with large values of C and which is a feature of many real networks [9, 10], gives rise to more fragile networks.

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