Universality of Two Dimensional Sandpiles

E. Goles[a], A. Gajardo[b1]

[a]Centre for Mathematical Modeling and Departamento de Ingeniería Matemática,
Universidad de Chile, UMR 2071-CNRS,
Casilla 170/3-Correo 3, Santiago, Chile

Institute for Complex Systems (ISCV),
Av. Artillería 600B, C^o Artillería, Valparaiso, Chile

[b]Departamento de Ingeniería Matemática,
Universidad de Concepción,
Casilla 160-C, Concepción, Chile

Abstract. We present a negative result for the Universality of Sandpiles in dimension two. Universality is taken in the sense of Banks -which consists into embed a logical circuit in the cellular space. We prove that in this context it is not possible to cross information, giving by this way a strong argument to say that Sandpiles are not Universal in dimension two; at least for the usual neighborhoods. Nevertheless, if a neighborhood of radius two is used, the Universality is possible, which is proved.

Key words: Sandpiles, complexity, Discrete Dynamical System, Cellular Automata, Calculability.

1 Introduction

Universality of dynamical systems is understood as the ability of "simulating" a Turing Machine. There is not agreement in the community on the meaning of "simulation", but many researchers agree that it must be strong enough to imply the existence of undecidable problems related with the dynamics of the system. The first universal Cellular Automaton (CA) was the one proposed by von Neumann [1], when the first formal definition of CAs was given. He defined Cellular Automata and showed how a universal and self reproducing Turing Machine can be emulated through it. He proved at the same time the computing ability of CAs and the existence of automatic systems with the ability of self reproduction, which was his actual goal.

Intended as the ability of computing, Universality was looked for in simple systems. Banks optimized the result of von Neumann defining a Universal CA in dimension two to only two states and the same neighborhood [2]. The smaller neighborhood for dimension two (of size three) was studied by Gajardo et al [3],

[1]Tel. (56-41) 207042, FAX. (56-41) 522055, anahi@ing-mat.udec.cl

where Universality of a three state CA is proved. Another particularly simple example of universal CA is the famous Game of Life [4]. But the simplest one is the elementary CA with rule 110, proved by Cook [5]. Other dynamical systems was studied from this point of view, see for example [6, 7].

In these days, Universality of a dynamical system is viewed as an index which talks about the predictability of the system, where predictability is measured under computer science scope. More precisely, the computational complexity of predicting whether a given pattern will or will not appear for a given initial configuration is studied. If the system is universal, this problem is intractable. This means that in order to know the behavior of the system there is not any method better than observing the system it self.

In order to simulate a Turing Machine with a CA -as the Sandpile,- the most used method consists in embedding an infinite logical circuit in the initial configuration in such a way that the logical circuit is computed when the system runs. The logical circuit is able to compute the evolution of the Turing Machine over some initial word. In this way, the problem of deciding whether some given pattern will or will not appear in the CA evolution when it starts over a given finitely described initial configuration is undecidable. This method was introduced by Banks. Embedding a logical circuit in a two dimensional CA is worked out by defining configurations that emulate wires which transport boolean values, logical gates, wire bifurcations and wire crossovers.

Sandpiles were defined by Bak et al in 1987 [8] as an example of a dynamical system presenting both power law distribution of *avalanches* size and a property which he named *self organized critically*. This property establishes that the system always evolves to a set of states (an attractor) which is critical in the sense that avalanches are frequent. Bak observed this for real piles of sand, and introduced a simplified discrete version with the same property, known as Sandpile and defined as follows.

Definition [2D-Sandpile] *Let $N \subset \mathbb{Z}^2$ be a finite set called neighborhood. The elements of $\mathbb{Z}^2$ will be called cells. The neighborhood of a cell i will be the set $\{i+j | j \in N\}$. A finite number of tokens, $x_i(0) \geq 0$, is assigned to each cell. For $t \geq 0$ the system evolves under the following rule applied synchronously to each cell.*

> *If $x_i(t)$ exceeds $|N|$, then the cell "fires" and all its neighbors increase its number of tokens by the number of times that $|N|$ divides $x_i(t)$, while the number of tokens of i decreases by $|N|$ times this number.*

Goles et al [9, 10, 11] studied Sandpiles from this approach. They shown that the system is Universal for an arbitrary graph of degree at most three. Moore et al [12] generalized this result to the sandpile over a three dimensional square grid. Banks's method cannot be applied for dimension one, and Moore asserted that logical circuits cannot be computed with a Sandpile in dimension one by using computational complexity arguments. Moore also conjectured that the same is true for dimension two, due to the apparent impossibility of the system of crossing information. In [13] we define formally what a wire crossover is and we prove that it cannot exist in a two dimensional Sandpile. In the following we describe the main steps of the proof.

The Crossover and the Firing Graph

A device as a logical gate or a Crossover is defined by a configuration over a finite portion of $\mathbb{Z}^2$. In order to study the Crossover one may consider, without loss of generality, that it is defined over a finite $n \times n$ square, i. e., it is an assignment of tokens to each cell of a $n \times n$ square.

A configuration is said to be *quiescent* if and only if each cell has strictly less than $|N|$ tokens. When a token is added to a cell of a quiescent configuration, the Sandpile evolves to a new quiescent configuration, in this case we say that an *avalanche* was produced.

A Crossover is defined as a configuration such that: 1) if a token is added to a given cell on its West border, an avalanche is produced, a token falls on its East border and no token falls on its South side. 2) the analogous happens when we add a token on a particular site on its North border.

The Crossover satisfies our intuition about crossing information in the sense that a token appears in the East side if and only if a token is added on a given cell of the West side, and a the analogous happens for the North to South direction.

An important property of configurations defined over finite regions is that if one token is added to the cells of the border, then each cell fires at most one time -if a token is added in the middle of the square it is possible that several cells fire several times. This allows us to introduce the following concept.

Definition 1 (Firing Graph) *Let us consider a quiescent configuration $c : \{1, .., n\}^2 \to \{0, .., 3\}$ and a cell (i, j) on the boundary of c. We define its Firing Graph (see Figure 1) as the directed graph $G = (V, E)$ where:*

V = the set of cells in $\{1, 2, .., n\} \times \{1, 2, .., n\}$ that fire if a token is added to the cell (i, j) in c, and
E is defined by $(u, v) \in E \Leftrightarrow u$ and v are neighbors and u fires before v.

Some direct properties of G are the following.

0	0	0	2	1	3
1	3	3	2	2	3
3	3	3	3	3	1
3	1	3	2	0	1
1	2	3	0	1	0

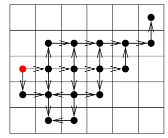

Figure 1: A configuration and its Firing Graph.

- It is connected.
- It has no cycles.
- Only the vertex (i,j) is a source of G (a vertex with in-degree equal to 0).
- If all the neighbors of u are in V then so is u.
- If the in-degree of u is k then u has at least $|N|-k$ tokens in the initial configuration (i.e., $c(u) \geq |N|-k$).

The main results

Theorem 1 *There does not exist a wire crossover for the Sandpile in $\mathbb{Z}^2$ with either a planar neighborhood or with the Moore neighborhood:* $\{(i,j)|\max\{|i|,|j|\} = 1\}$.

The idea of the proof consists in to show that it is possible to suppose that the Firing Graphs associated to the North-South and West-East avalanches G_{ns} and G_{we} respectively are vertex disjoint. This immediately shows the impossibility of the crossover over a planar grid. For the Moore neighborhood we observe that if a vertex of G_{ns} do not fire during the West-East avalanche, then it has more predecessors in G_{ns} than neighbors in G_{we}. But this property makes impossible the connexity of both graphs, carring to a contradiction.

This theorem discards the Banks method for proving the universality of this system. But it do not prove decidability of problems associated with 2D-Sandpiles. On the other hand, if we consider a neighborhood of radius two, the Sandpile becomes universal as the following theorem establishes.

Theorem 2 *The Sandpile over $\mathbb{Z}^2$ with the von Neumann neighborhood of radius $k \geq 2$*
($N = \{0\} \times \{-k,..,-1,1,..,k\} \cup \{-k,..,-1,1,..,k\} \times \{0\}$) is Universal.

The proof consists in to define the basic devices that are used to construct logical circuits. Figure 2 shows the devices for this automaton.

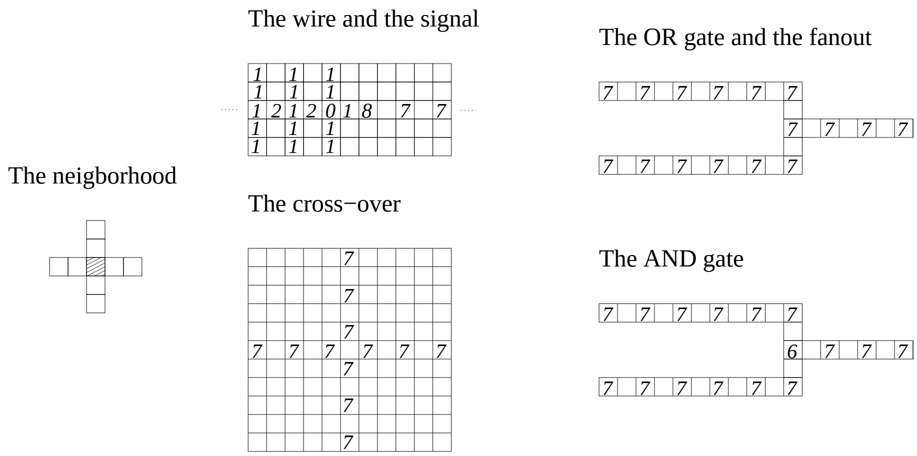

Figure 2: The devices that show the Universality of the Sandpile on a von Neumann neighborhood of radius 2. The wire is a path of cells with seven tokens at distance two. The signal is a cell with eight tokens. When the signal propagates, the wire is distorted.

Open questions

In this work we suppose that a crossover is emulated by a finite configuration which needs to be stable, and that signals are perturbations that propagates over a stable background.

It may be possible to conceive signals propagating over an unstable and periodic background. In this context we could define the different devices as finite patterns which do not interact with the periodical environment in the following way. A set of finite patterns: $\{d_i : A_i \to S\}_{i=1}^{n}$ is a *set of devices* if and only if there exists a periodic configuration $p : \mathbb{Z}^2 \to S$ such that for every i, the configuration g_i defined by $g_i(x) = d_i(x)$ if $x \in A_i$ and $p(x)$ in other case, satisfies that $T^k(g_i)|_{\mathbb{Z}^2 \setminus A_i} = T^k(p)|_{\mathbb{Z}^2 \setminus A_i}$ for every $k \in \mathbb{N}$.

On the other hand, the impossibility of crossing information is an obstacle To apply the Banks method, but it is not known whether it is an obstacle for Universality it self. We think that it is in fact an impediment, but a formal proof of this may require mathematical tools that are not yet developed.

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